Local analytic geometry, II. Theory of finite morphisms

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1 Analytic spectrum of an algebra of finite presentation

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Definition 1. Let S be a ringed space. We say that an $\mathscr{O}_S$-algebra $\mathscr{A}$ is of *finite presentation* if every $s \in S$ admits a neighbourhood U such that $\mathscr{A}|U$ is isomorphic to a sheaf of the form

$$\frac{(\mathscr{O}_S|U)[t_1,\ldots,t_n]}{(f_1,\ldots,f_m)}$$

with $f_1,\ldots,f_m \in \Gamma(U,\mathscr{O}_S[t_1,\ldots,t_n])$.

That is, $\mathscr{A}$ is locally generated over $\mathscr{O}_S$ by a finite number of sections subject to a finite number of relations.

We are interested in the case where S is an analytic space, over a complete valued non-discrete base field k. For every analytic space T over S, consider the sheaf $\mathscr{A}_T = \mathscr{A} \times_S T$, the inverse image of $\mathscr{A}$ on T, and the set $F_{\mathscr{A}}(T) = \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{A}_T,\mathscr{O}_T)$ of algebra homomorphisms; we immediately see that $T \mapsto F_{\mathscr{A}}(T)$ defines a contravariant functor $F_{\mathscr{A}}\colon (\mathsf{An})^{\mathrm{op}}/S \to \mathsf{Set}$.

Proposition . *Let $\mathscr{A}$ be an algebra of finite presentation on an analytic space S; the functor $F_{\mathscr{A}}$ is representable in the category An/S by an analytic space separated over S.*

Proof. We note first of all that the functor $F_{\mathscr{A}}$ is of a local nature (cf. Exposé 11, Definition 5.4), since if T is an analytic space over S, then the presheaf $U \mapsto F_{\mathscr{A}}(U)$ (where U runs over the opens of T) is precisely the sheaf of germs of algebra homomorphisms $\underline{\mathrm{Hom}}_{\mathscr{O}_T}(\mathscr{A}_T, \mathscr{O}_T)$.

Thus, by Corollary 5.7 of Exposé 11, it suffices, to show that $F_{\mathscr{A}}$ is representable, to find an open cover (S_i) of S such that, for each index i, the functor $F_{\mathscr{A}}/S_i$ is representable in An/S_i.

By Definition 1, we thus see that it suffices to consider the case where $\mathscr{A}$ is of the form $\mathscr{O}_S[t_1,\ldots,t_n]/(f_1,\ldots,f_m)$. Then the exact sequence p. 19-02

$$0 \to \mathscr{I} \xrightarrow{i} \mathscr{O}_S[t_1,\ldots,t_n] \to \mathscr{A} \to 0$$

(where $\mathscr{I} = (f_1,\ldots,f_m)$) gives, by base extension, an exact sequence

$$\mathscr{I}_T \xrightarrow{i_T} \mathscr{O}_T[t_1,\ldots,t_n] \to \mathscr{A}_T \to 0$$

for every analytic space T over S. This allows us to identify $F_{\mathscr{A}}(T) = \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{A}_T, \mathscr{O}_T)$ with the set of homomorphisms from $\mathscr{O}_T[t_1,\ldots,t_n]$ to $\mathscr{O}_T$ that vanish on $i_T(\mathscr{I}_T)$.

But the functor $F_{\mathscr{O}_S[t_1,\ldots,t_n]} \colon T \mapsto \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{O}_T[t_1,\ldots,t_n], \mathscr{O}_T)$ is isomorphic to $T \mapsto (\Gamma(T,\mathscr{O}_T))^n$; by Theorem 1.1 of Exposé 10 and Proposition 3.1 of Exposé 11, we see that $F_{\mathscr{O}_S[t_1,\ldots,t_n]}$ is represented in An/S by the pair $(S \times \mathscr{E}^n, \eta)$, where η is the homomorphism of algebras $\eta\colon \mathscr{O}_{S\times\mathscr{E}^n}[t_1,\ldots,t_n] \to \mathscr{O}_{S\times\mathscr{E}^n}$ defined by $\eta(t_i) = q^*(z_i)$ (for $i = 1,\ldots,n$) and denoting by q the projection from $S \times \mathscr{E}^n$ onto $\mathscr{E}^n$. It immediately follows that the sub-functor $F_{\mathscr{A}}$ is represented by the closed subspace $X \subset S \times \mathscr{E}^n$ defined by the ideal $\mathscr{I} = \eta \circ i_{S\times\mathscr{E}^n}(\mathscr{I}_{S\times\mathscr{E}^n}) \subset \mathscr{O}_{S\times\mathscr{E}^n}$ (cf. Exposé 11, Lemma 3.6).

In the case in question, X is evidently separated over S. Since the property of being separated over S, for an object of An/S, is local on S, the latter claim of the statement is also proven (cf. Expose 11, Proposition 5.6: we obtain the space that represents $F_{\mathscr{A}}$ by gluing together those that represent the $F_{\mathscr{A}/S_i}$). ☐

Definition 2. For every $\mathscr{O}_S$-algebra $\mathscr{A}$ of finite presentation, we define the *analytic spectrum* of $\mathscr{A}$, denoted $\mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$, to be the object of An/S (defined up to S-isomorphism) that represents the functor

$$F_{\mathscr{A}} \colon T \longmapsto \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{A}_T, \mathscr{O}_T).$$

$\mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$ is thus an analytic space separated over S. p. 19-3

Example. Consider an $\mathscr{O}_S$-module $\mathscr{E}$ of finite presentation; its symmetric algebra $S(\mathscr{E})$ is an $\mathscr{O}_S$-algebra of finite presentation, and $\mathrm{Spec}_{\mathrm{an}}(S(\mathscr{E}))$ represents the functor

$$T \longmapsto \mathrm{Hom}_{\mathscr{O}_T\text{-}\mathsf{Alg}}(S(\mathscr{E}_T), \mathscr{O}_T) \simeq \mathrm{Hom}_{\mathscr{O}_T\text{-}\mathsf{Mod}}(\mathscr{E}_T, \mathscr{O}_T).$$

This is the *vector bundle* defined by $\mathscr{E}$ (cf. Exposé 12, §1).

Proposition 2.

i. *$\mathscr{A} \mapsto \mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$ defines a contravariant functor from the category of $\mathscr{O}_S$-algebras of finite presentation to* An/S.
ii. *For any $\mathscr{O}_S$-algebras $\mathscr{A}$ and $\mathscr{B}$ of finite presentation, $\mathscr{A} \otimes_{\mathscr{O}_S} \mathscr{B}$ is of finite presentation, and* $\mathrm{Spec}_{\mathrm{an}}(\mathscr{A} \otimes_{\mathscr{O}_S} \mathscr{B}) \simeq \mathrm{Spec}_{\mathrm{an}}(\mathscr{A}) \times_S \mathrm{Spec}_{\mathrm{an}}(\mathscr{B})$.

iii. *Let $h\colon \mathscr{A} \to \mathscr{B}$ be a surjective homomorphism of algebras of finite presentation. Then the corresponding morphism* $\mathrm{Spec}_{\mathrm{an}}(h)\colon \mathrm{Spec}_{\mathrm{an}}(\mathscr{B}) \to \mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$ *is a closed immersion.*
iv. *For every $\mathscr{O}_S$-algebra $\mathscr{A}$ of finite presentation, and for every analytic space T over S,* $\mathrm{Spec}_{\mathrm{an}}(\mathscr{A}) \times_S T \simeq \mathrm{Spec}_{\mathrm{an}}(\mathscr{A}_T)$ *(compatibility with base change).*

Proof.

(i) is immediate, since $F_{\mathscr{A}}(T)$ is a contravariant functor in $\mathscr{A}$;
(ii) follows from the formula

$$\mathrm{Hom}_{\mathscr{O}_T}(\mathscr{A}_T \otimes_{\mathscr{O}_T} \mathscr{B}_T, \mathscr{O}_T) \simeq \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{A}_T, \mathscr{O}_T) \times \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{B}_T, \mathscr{O}_T)$$

(iii) follows easily from the proof of Proposition 1, and (iv) can be proven by Corollary 3.2 of Exposé 11.

□

Consider the pair (X,ξ) that represents the functor $F_{\mathscr{A}}$ defined by an $\mathscr{O}_S$-algebra $\mathscr{A}$ of finite presentation. Then $X = \mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$ is an analytic space over S, with structure morphism $f\colon X \to S$, and $\xi \in F_{\mathscr{A}}(X) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{A}_X, \mathscr{O}_X)$ is a homomorphism from $\mathscr{A}_X = f^*(\mathscr{A})$ to $\mathscr{O}_X$. In other words, ξ defines a factorisation

$$X \xrightarrow{\hat{f}} (|S|, \mathscr{A}) \to S$$

of the structure morphism f of X through the ringed space $(|S|,\mathscr{A})$ (where $|S|$ denotes the underlying topological space of S) endowed with the canonical morphism $(|S|,\mathscr{A}) \to S$ (consisting of the identity map on $|S|$ and the homomorphism $\mathscr{O}_S \to \mathscr{A}$). *p. 19-04*

For any $\mathscr{A}$-module $\mathscr{M}$, set

$$\widetilde{\mathscr{M}} = \hat{f}^*(\mathscr{M}) = f^*(\mathscr{M}) \otimes_{f^*(\mathscr{A})} \mathscr{O}_X.$$

We thus define a functor $\mathscr{M} \mapsto \widetilde{\mathscr{M}}$ from the category of $\mathscr{A}$-modules to the category of $\mathscr{O}_X$-modules (where $X = \mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$). If $\mathscr{M}$ and $\mathscr{N}$ are $\mathscr{A}$-modules then

$$(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})^{\sim} = \widetilde{\mathscr{M}} \otimes_{\mathscr{O}_X} \widetilde{\mathscr{N}}.$$

Consider an analytic space T over S, and the space

$$X_T = \mathrm{Spec}_{\mathrm{an}}(\mathscr{A}_T) = X \times_S T$$

over T ((iv) of Proposition 2); the structure morphism from X_T to T again factors as

$$X_T \xrightarrow{\hat{f}_T} (|T|, \mathscr{A}_T) \to T$$

and we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\hat{f}"'] & X_T \arrow[l] \arrow[d, "\hat{f}_T"] \\ (|S|, \mathscr{A}) & (|T|, \mathscr{A}_T) \arrow[l] \end{tikzcd}$$

Let $\mathscr{M}$ be an $\mathscr{A}$-module; the inverse image $\mathscr{M}_T = \mathscr{M} \times_S T$ of $\mathscr{M}$ over T is an $\mathscr{A}_T$-module. We thus see that $\widetilde{\mathscr{M}_T} = \hat{f}_T^*(\mathscr{M}_T)$ can be identified with the inverse image of $\widetilde{\mathscr{M}}$ under the morphism $X_T \to X$ (compatibility of the functor $\mathscr{M} \mapsto \widetilde{\mathscr{M}}$ with base change).

2 Fibre of the analytic spectrum over a point in the base

Consider an $\mathscr{O}_S$-algebra $\mathscr{A}$ of finite presentation, and its analytic spectrum $X = \mathrm{Spec}_{\mathrm{an}}(\mathscr{A})$. If $s \in S$, recall (Exposé 10, §2) that the *fibre* of X at s is the fibre product $X_s = X \times_S \{s\}$, where $\{s\}$ is the analytic space consisting of the point s, endowed with the field $\mathscr{O}_s/\mathfrak{m}_s\mathscr{O}_s = K(s) \simeq k$ (where $\mathfrak{m}_s$ denotes the maximal ideal of the local ring $\mathscr{O}_s$). Then (by (iv) of Proposition 2) p. 19-05

$$X_s = X \times_S \{s\} \simeq \mathrm{Spec}_{\mathrm{an}}(\mathscr{A}_{\{s\}}) = \mathrm{Spec}_{\mathrm{an}}(\mathscr{A}(s))$$

where $\mathscr{A}(s) = \mathscr{A}_s/\mathfrak{m}_s\mathscr{A}_s = \mathscr{A}_s \otimes_{\mathscr{O}_s} k = \mathscr{A}_{\{s\}}$; if on a neighbourhood U of s we have that

$$\mathscr{A}|U \simeq \frac{\mathscr{O}_U[t_1,\ldots,t_n]}{(f_1,\ldots,f_m)}$$

then

$$\mathscr{A}(s) \simeq \frac{k[t_1,\ldots,t_n]}{(f_1(s),\ldots,f_m(s))}.$$

$X_s = \mathrm{Spec}_{\mathrm{an}}(\mathscr{A}(s))$ is the closed subspace of $\{s\} \times \mathscr{E}^n \simeq \mathscr{E}^n$ defined by the "equations"

$$f_i(s)(z_1,\ldots,z_n) = 0 \qquad \text{for } i = 1,2,\ldots,m.$$

More precisely, if the functor $F_{\mathscr{A}}$ is represented in An/S by the pair (X,ξ), then the functor $F_{\mathscr{A}(s)} = F_{\mathscr{A}/\{s\}}$ is represented in $\mathsf{An}/\{s\}$ by the pair (X_s,ξ_s), where X_s is the fibre of X at s and where $\xi_s\colon \mathscr{A}_{X_s} \to \mathscr{O}_{X_s}$ is obtained from $\xi\colon \mathscr{A}_X \to \mathscr{O}_X$ by reduction modulo $\mathfrak{m}_s$. In particular, the map from $\mathrm{Hom}_{\{s\}}(\{s\},X_s)$ to $\mathrm{Hom}_k(\mathscr{A}(s),k)$ that sends a section f of $X_s/\{s\}$ to the composite homomorphism $f^*\circ\xi'_{s,f(s)}$ (where $\xi'_{s,f(s)}\colon \mathscr{A}(s) \to \mathscr{O}_{X_s,f(s)}$ is induced by the germ of ξ_s at $f(s)$) is bijective. Since $\mathrm{Hom}(\{s\},X_s)$ can be identified with the set of points of X_s by sending $f:\{s\} \to X_s$ to the point $f(s) = x$ (so that $f^* = \epsilon_X\colon \mathscr{O}_{X_s,x} \to k$ is the augmentation homomorphism), we obtain a bijection between the set of points of X_s and the set of ideals of $\mathscr{A}(s)$ that give k as their quotient, by sending $x \in X_s$ to the inverse image under $\xi'_{s,x}\colon \mathscr{A}(s) \to \mathscr{O}_{X_s,x}$ of the maximal ideal of $\mathscr{O}_{X_s,x}$. But $\mathscr{O}_{X_s,x} = \mathscr{O}_{X,x}/\mathfrak{m}_s\mathscr{O}_{X,x}$ (Exposé 10, §2), $\mathscr{A}(s) = \mathscr{A}_s/\mathfrak{m}_s\mathscr{A}_s$, and $\xi'_{s,x}$ is obtained by reduction modulo $\mathfrak{m}_s$ of $\xi'_X\colon \mathscr{A}_s \to \mathscr{O}_{X,x}$ (induced by the germ of ξ at x). We can thus state: p. 19-06

Proposition 3. *We*