The primary obstruction to deformation

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Introduction

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Let V_0 be a compact complex-analytic manifold, and let Θ be the sheaf of germs of holomorphic fields of tangent vectors. We ask the following question: given an element $a \in \mathrm{H}^1(V_0,\Theta)$, does there exists a deformation of V_0, with a non-singular base (i.e. a fibred mixed manifold $\pi\colon V \to B$, with $b_0 \in B$, along with an isomorphism $V_0 \xrightarrow{\cong} \pi^{-1}(b_0)$), such that a is the image, under the map ρ defined in [Talk no. 2], of a vector v that is tangent to B at b_0? An element $a \in \mathrm{H}^1(V_0,\Theta)$ for which the answer is positive is called a *deformation vector*. We will give a necessary condition for a to be a deformation vector; this condition is written $[a \smile a] = 0$. We will then give an example where this condition is not satisfied.

I. Exact sequences of sheaves of algebras

Let K be a commutative ring, and let Φ, Φ_1, and Φ_2 be sheaves of K-modules on some space X, and suppose that we have some given homomorphism $\Phi_1 \otimes \Phi_2 \to \Phi$, written as a product. We define, for any cover $\mathscr{U}$ of X, the *cup product*

$$\smile\colon C^p(X,\mathscr{U};\Phi_1) \otimes C^q(X,\mathscr{U};\Phi_2) \to C^{p+q}(X,\mathscr{U};\Phi)$$

by the formula

$$(\alpha \smile \beta)_{i_0,\ldots,i_{p+q}} = \alpha_{i_0,\ldots,i_p} \cdot \beta_{i_p,\ldots,i_{p+q}}.$$

We have the relation

$$\mathrm{d}(\alpha \smile \beta) = \mathrm{d}\alpha \smile \beta + (-1)^p \alpha \smile \mathrm{d}\beta.$$

This induces a cup product on the cohomology of the cover $\mathscr{U}$, and, by passing to the inductive limit over open covers, a cup product

$$\smile\colon \mathrm{H}^p(X;\Phi_1) \otimes \mathrm{H}^q(X;\Phi_2) \to \mathrm{H}^{p+q}(X;\Phi).$$

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Definition. A *sheaf of algebras* on X is a sheaf of modules Φ on X endowed with a product $\Phi \otimes \Phi \to \Phi$ (which we do not assume to be either commutative nor associative).

If $f\colon \Phi \to \Psi$ is a homomorphism of sheaves of algebras, then the kernel Φ' of f is a sheaf of two-sided ideals of Φ, i.e. we have products $\Phi' \otimes \Phi \to \Phi'$ and $\Phi \otimes \Phi' \to \Phi'$ such that the two diagrams

$$\begin{tikzcd} \Phi' \otimes \Phi \arrow[r] \arrow[d] & \Phi' \arrow[d] \\ \Phi \otimes \Phi \arrow[r] & \Phi \end{tikzcd} \qquad \begin{tikzcd} \Phi \otimes \Phi' \arrow[r] \arrow[d] & \Phi' \arrow[d] \\ \Phi \otimes \Phi \arrow[r] & \Phi \end{tikzcd}$$

both commute.

Proposition 1. *Let $0 \to \Phi' \to \Phi \to \Phi'' \to 0$ be an exact sequence of sheaves of algebras on X; let $a \in \mathrm{H}^p(X;\Phi'')$. Then $\delta a \in \mathrm{H}^{p+1}(X;\Phi')$, and, for any class $b \in \mathrm{H}^q(X;\Phi')$, we have $\delta a \smile b = 0$.*

Proof. Let $\mathscr{U}$ be a cover of X such that a and b are represented by cocycles α and β (respectively), and such that α lifts to a cochain $\eta \in C^p(X,\mathscr{U};\Phi)$. Then $\delta\eta$ is a cocycle in $C^{p+1}(X,\mathscr{U};\Phi')$ whose class in $\mathrm{H}^{p+1}(X;\Phi')$ is, by definition, δa, and $\delta a \smile b$ is the class of $\delta\eta \smile \beta$. But $\delta(\eta \smile \beta) = \delta\eta \smile \beta$, and $\eta \smile \beta$ is a cochain in $C^{p+q}(X,\mathscr{U};\Phi')$, since Φ' is a sheaf of ideals. So the cocycle $\delta\eta \smile \beta$ is cohomologous to 0 in $\mathrm{H}^{p+q+1}(X;\Phi')$, which proves the proposition. $\square$

II. The primary obstruction

Let V_0 be a complex-analytic manifold, and Θ_0 the sheaf of germs of holomorphic fields of tangent vectors. Then Θ_0 is a sheaf of Lie algebras, and, if $a, b \in \mathrm{H}^\bullet(V_0, \Theta_0)$, then we denote by $[a \smile b]$ the cup product defined by the bracket $[-,-]\colon \Theta_0 \otimes \Theta_0 \to \Theta_0$. It satisfies

$$[b \smile a] = (-1)^{pq+1}[a \smile b]$$

for $a \in \mathrm{H}^p(V_0, \Theta_0)$ and $b \in \mathrm{H}^q(V_0, \Theta_0)$. *p. 4-03*

Theorem 1. *Let $\pi\colon V \to B$ be a mixed manifold, b_0 a point of B, $V_0 = \pi^{-1}(b_0)$, and let $\rho_0\colon T_0 \to \mathrm{H}^1(V_0, \Theta_0)$ be Spencer–Kodaira map. Then, if u and v are tangent vectors of B at b_0, we have*

$$[\rho_0(u) \smile \rho_0(v)] = 0.$$

Corollary. *Let V_0 be a complex-analytic manifold, and Θ the sheaf of germs of holomorphic fields of tangent vectors of V_0. If $a \in \mathrm{H}^1(V_0, \Theta)$ is a deformation vector, then*

$$[a \smile a] = 0.$$

Proof. (Proof of the Corollary). This is simply a particular case of Theorem 1; note that $[a \smile b]$ is a symmetric bilinear map from $\mathrm{H}^1 \otimes \mathrm{H}^1$ to H^2, and that we are in characteristic $0 \neq 2$. □

Proof. (Proof of Theorem 1). Consider the following sheaves on V_0:

- Θ_0: the sheaf of germs of vertical holomorphic fields on V_0;
- $\widetilde{\Theta}_0$: the sheaf of germs of vertical holomorphic fields on V;
- Π_0: the sheaf of germs of locally projectable holomorphic fields on V_0;
- $\widetilde{\Pi}_0$: the sheaf of germs of locally projectable holomorphic fields on V;
- Λ_0: the sheaf $\pi^* T_0$, where T_0 is the tangent space of B at b_0; and
- $\widetilde{\Lambda}_0$: the sheaf $\pi^* \widetilde{T}_0$, where $\widetilde{T}_0$ is the space of germs at b_0 of fields on B of tangent vectors of B.

We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\Theta}_0 \arrow[r] \arrow[d, "\varepsilon"'] & \widetilde{\Pi}_0 \arrow[r] \arrow[d, "\varepsilon"'] & \widetilde{\Lambda}_0 \arrow[r] \arrow[d, "\varepsilon"'] & 0 \\
0 \arrow[r] & \Theta_0 \arrow[r] & \Pi_0 \arrow[r] & \Lambda_0 \arrow[r] & 0
\end{tikzcd}$$

whence we obtain the following commutative diagram: *p. 4-04*

$$\begin{tikzcd}
\widetilde{T}_0 \arrow[r, "\widetilde{\rho}"] \arrow[d, "\varepsilon"'] & \mathrm{H}^1(V_0; \widetilde{\Theta}) \arrow[d, "\varepsilon"] \\
T_0 \arrow[r, "\rho"'] & \mathrm{H}^1(V_0; \Theta_0)
\end{tikzcd}$$

Let $u, v \in T_0$ be fixed tangent vectors of B at b_0. We can always find vector fields $\widetilde{u}$ and $\widetilde{v}$ on B that take the values u and v (respectively) at b_0; $\epsilon(\widetilde{u}) = u$ and $\epsilon(\widetilde{v}) = v$. The exact sequence

$$0 \to \widetilde{\Theta}_0 \to \widetilde{\Pi}_0 \to \widetilde{\Lambda}_0 \to 0$$

is a sequence of homomorphisms of sheaves of Lie algebras, and so

$$[\widetilde{\rho}(\widetilde{u}) \smile \widetilde{\rho}(\widetilde{v})] = 0$$

by Proposition 1. But $\epsilon\colon \widetilde{\Theta}_0 \to \Theta_0$ is also a homomorphism of sheaves of Lie algebras, and the diagram

$$\begin{tikzcd}
\mathrm{H}^1(V_0,\widetilde{\Theta}_0)\otimes \mathrm{H}^1(V_0,\widetilde{\Theta}_0) \arrow[r, "{[-\smile-]}"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & \mathrm{H}^2(V_0,\widetilde{\Theta}_0) \arrow[d, "\varepsilon"] \\
\mathrm{H}^1(V_0,\Theta_0)\otimes \mathrm{H}^1(V_0,\widetilde{\Theta}_0) \arrow[r, "{[-\smile-]}"'] & \mathrm{H}^2(V_0,\Theta_0)
\end{tikzcd}$$

commutes. We thus deduce that $[\rho(u) \smile \rho(v)] = 0$. □

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Remarks.

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1. We make essential use of the fact that $\epsilon\colon \widetilde{T}_0 \to T_0$ is surjective, and thus of the fact that B has no singularities.
2. We actually have $[\rho(u) \smile b] = 0$ for all $u \in T_0$, for any class $b \in \mathrm{H}^1(V_0,\Theta_0)$ that is in the image of $\mathrm{H}^1(V_0,\widetilde{\Theta}_0)$ under ϵ. In particular, for an element $a \in \mathrm{H}^1(V_0,\Theta_0)$ to be a regular deformation vector (in the sense of [Talk no. 3]), it is necessary and sufficient for $[a \smile b] = 0$ for all $b \in \mathrm{H}^1(V_0,\Theta_0)$.

If V_0 is a compact complex-analytic manifold, and $a \in \mathrm{H}^1(V_0,\Theta)$, then we call $[a \smile a] \in \mathrm{H}^2(V_0,\Theta)$ the *primary obstruction* to the deformation of V_0 along a. For a to be a deformation vector, it is necessary that this primary obstruction be zero; but it is not sufficient: we can define a sequence of set-theoretic maps ω_n, called *obstructions*, with $\omega_1\colon \mathrm{H}^1(V_0,\Theta) \to \mathrm{H}^2(V_0,\Theta)$ given by $\omega_1(a) = [a \smile a]$, and with ω_{k+1} defined on the subset of $\mathrm{H}^1(V_0,\Theta)$ where ω_k vanishes, with values in varying quotients[1] of $\mathrm{H}^2(V_0,\Theta)$, and a necessary condition for a to be a deformation vector is that all the $\omega_k(a)$ be defined and real. I do not know if *this* condition is sufficient. Kodaira, Spencer, and Nijenhuis [4] have shown that, if $\mathrm{H}^2(V_0,\Theta) = 0$, then every element of $\mathrm{H}^1(V_0,\Theta)$ is a deformation vector. In this case, we even have a locally universal deformation whose base is a manifold, and ρ is an isomorphism from the tangent space of this manifold to $\mathrm{H}^1(V_0,\Theta)$

III. An example of obstruction

1. The manifold V_0

Let $X = E/\Gamma$ be a 2-dimensional complex torus, i.e. $E \cong \mathbb{C}^2$ and $\Gamma \cong \mathbb{Z}^4$, and let D the be projective line $\mathbb{P}^1\mathbb{C}$. Set $V_0 = X \times D$. The sheaf Θ of holomorphic fields of tangent vectors of V_0 is the direct sum of the sheaves of Lie algebras Θ_1 and Θ_2, where

$$\Theta_1 = \mathscr{O} \otimes_{\mathscr{O}_X} \pi_1^*\Theta_X$$
$$\Theta_2 = \mathscr{O} \otimes_{\mathscr{O}_D} \pi_2^*\Theta_D$$

where $\pi_1\colon V_0 \to X$ and $\pi_2\colon V_0 \to D$ are the projections, $\mathscr{O}$, $\mathscr{O}_X$, and $\mathscr{D}$ are the structure sheaves (sheaves of local rings), and Θ_X and Θ_D are the sheaves of germs of holomorphic fields of

tangent vectors of X and D (respectively). We are mostly interested in Θ_2. Also, $\mathrm{H}^1(V_0,\Theta_2)$ is *p. 4-06*
given by the Künneth exact sequence:

$$0 \to \mathrm{H}^0(X,\mathcal{O}_X)\otimes \mathrm{H}^1(D,\Theta_D) \to \mathrm{H}^1(V_0,\Theta_2) \to \mathrm{H}^1(X,\mathcal{O}_X)\otimes \mathrm{H}^0(D,\Theta_D) \to 0.$$

But we know that $\mathrm{H}^0(D,\Theta_D)$ is the Lie algebra $\mathfrak{a}$ of the group

$$A = \mathrm{GL}(2,\mathbb{C})/\mathbb{C}^* = \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$$

of automorphisms of D, and that $\mathrm{H}^1(D,\Theta_D) = 0$, as we can easily see by taking a cover of D by two open subsets. We have already seen (in [Talk no. 1]) that, if $X = E/\Gamma$, then $\mathrm{H}^1(X,\mathcal{O}) = \mathrm{Hom}(\Gamma,\mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E,\mathbb{C})$ is of dimension 2. So $\mathrm{H}^1(V_0,\Theta_2) = \mathrm{H}^1(X,\mathcal{O})\otimes\mathfrak{a}$ is of dimension 6. The cup product

$$\mathrm{H}^1(V_0,\Theta_2)\otimes \mathrm{H}^1(V_0,\Theta_2) \to \mathrm{H}^2(V_0,\Theta_2)$$

is given by the formula

$$[(\gamma\otimes\alpha)\smile(\gamma'\otimes\alpha')] = (\gamma\smile\gamma')\otimes[\alpha,\alpha'].$$

The cone of elements $\varphi \in \mathrm{H}^1(V_0,\Theta_2)$ such that $[\varphi\smile\varphi] = 0$ can be identified with the cone of rank 1 tensors in $\mathrm{H}^1(X,\mathcal{O})\otimes\mathfrak{a}$. Indeed, if $\varphi = \gamma\otimes\alpha$, then

$$[\varphi\smile\varphi] = (\gamma\smile\gamma)\otimes[\alpha,\alpha] = 0\otimes 0 = 0$$

and, if φ is not a simple tensor, then we have

$$\varphi = \gamma\otimes\alpha + \gamma'\otimes\alpha'$$

with γ and γ' independent, and α and α' independent, so

$$[\varphi\smile] = 2(\gamma\smile\gamma')\otimes[\alpha,\alpha'] \neq 0.$$

2. The mixed space V

In this example, every element of $\mathrm{H}^1(V_0,\Theta_2)$ whose primary obstruction is zero is a deformation vector. More precisely: *p. 2-07*

Proposition 2.
There exists a mixed space $\pi\colon V \to B$ and a point $b_0 \in B$ such that

1. *$\pi^{-1}(b_0) = V_0$ (the manifold defined in §III.1);*
2. *there exists an isomorphism σ from a $\mathbb{C}$-analytic space B to the cone of elements $\varphi \in \mathrm{H}^1(V_0,\Theta_2)$ such that $[\varphi\smile\varphi] = 0$; and*
3. *for every subspace B' of B that has no singularities at b_0, the Spencer–Kodaira map ρ from the tangent space of B' at b_0 to $\mathrm{H}^1(V_0,\Theta)$ agrees with $\sigma\colon B' \to \mathrm{H}^1(V_0,\Theta_2)$.*

Let H be the analytic space of homomorphisms from Γ to $\mathfrak{a}$ whose images are contained in a vector subspace of $\mathfrak{a}$ that is 1-dimensional over $\mathbb{C}$ (i.e. (4×2) matrices of rank 1 with coefficients in $\mathbb{C}$). For every $h \in H$, $e\circ h$ is a homomorphism from Γ to A, where $e\colon\mathfrak{a}\to A$ denotes the exponential map, and we construct a manifold V_h that is fibred over X with fibre D as follows: V_h is the quotient of $E\times D$ by the equivalence relation defined by Γ acting via

$$\gamma\star(x,y) = (x+\gamma,((e\circ h)(\gamma))\cdot y).$$

[1] See the Appendix.

These manifolds are the fibres of a mixed space $W \to H$, where W is the quotient of $H \times E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (h,x,y) = (h, x+y, (e \circ h(y)) \cdot y).$$

We now place the following equivalence relation on H: we have $h' \sim h$ if and only if $(h'-h)$ extends to an $\mathbb{C}$-linear map $f: E \to \mathfrak{a}$. Note that, if $h'(\Gamma)$ and $h(\Gamma)$ are contained in the same subspace L of $\mathfrak{a}$ of dimension 1 over $\mathbb{C}$ (or if $h' \sim h$), then we also have $f(E) \subset L$ (or $h \sim 0$ and $h' \sim 0$). In both cases, V_h and $V_{h'}$ are isomorphic, and we have an isomorphism $i_{h',h}: V_h \to V_{h'}$ defined by

$$i_{h',h}(x,y) = (x, e \circ f(x) \cdot y)$$

(in the first case), or

$$i_{h',h} = i_{h',0} \circ i_{0,h}$$

(in the second case). If h, h', and h'' are in the same class, then we have $i_{h''h} = i_{h''h'} \circ i_{h'h}$, and we can place on W the equivalence relation p. 4-08

$$(h',z') \sim (h,z) \iff h' \sim h \text{ or } z' = i_{h'h} z$$

for $h, h' \in H$, $z \in V_h$, and $z' \in V_{h'}$.

Let B and V be the quotients of H and W (respectively) by these equivalence relations. We have a projection $V \to B$. To show that the structures of a $\mathbb{C}$-analytic space on H and W induce structures of a $\mathbb{C}$-analytic space on their quotients B and V, it suffices to remark that we can lift B to a analytic subspace of H: let, for example, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be a basis of Γ such that (γ_1, γ_2) is a basis of E over $\mathbb{C}$; then each class $b \in B$ contains exactly one element $h \in H$ such that

$$h(\gamma_1) = h(\gamma_2) = 0.$$

3. Calculating ρ_0

Let T be the Zariski tangent space of B at b_0, i.e. the dual of $\mathfrak{I}/\mathfrak{I}^2$, where $\mathfrak{I}$ is the ideal of germs at b_0 of analytic functions on B that are zero at b_0. Then T_0 can be identified with $\mathrm{Hom}(\Gamma, a)/\mathrm{Hom}_{\mathbb{C}}(E, a)$. Also,

$$\begin{aligned} \mathrm{H}^1(V_0, \Theta) &= \mathrm{H}^1(V_0; \Theta_1) \oplus \mathrm{H}^1(V_0; \Theta_2) \\ &= \big(\mathrm{H}^1(X; \mathscr{O}) \otimes E\big) \oplus \big(\mathrm{H}^1(X; \mathscr{O}) \otimes a\big), \end{aligned}$$

and the second term of this term can be identified with the quotient $\mathrm{Hom}(\Gamma, a)/\mathrm{Hom}_{\mathbb{C}}(E, a)$. We are going to show that the map $\rho_0: T_0 \to \mathrm{H}^1(V_0; \Theta)$ is exactly the canonical injection defined by these identifications.

Let $u \in T_0 = \mathrm{Hom}(\Gamma, \alpha)/\mathrm{Hom}(E, \alpha)$ be the class of an element $h \in \mathrm{Hom}(\Gamma, \alpha)$, which we suppose to be of rank 1. Then we can write h in the form $\eta \otimes \sigma$, where $\eta \in \mathrm{Hom}(\Gamma, \mathbb{C})$, $\sigma \in \alpha$, and we can consider h as a tangent vector to H at 0. Let $\overline{h}$ be the field of tangent vectors to $H \times E \times D$ at $0 \times E \times D$ that projects onto h, and thus whose components over $E \times D$ are zero. Let (U_i) be a cover of $X = E/\Gamma$ by simply connected open subsets, and choose, for each i, a component $\widetilde{U}_i$ of the inverse image of U_i in E. We will denote by v_i the image over $U_i \times D$ of the field $\overline{h}|\widetilde{U}_i \times D$. This is a projectable holomorphic field on $0 \times U_i \times D$ of tangent vectors of $H \times U_i \times D$, and we set $w_{ij} = v_j - v_i$, so that w_{ij} is a vertical holomorphic field on $U_{ij} \times D$, and these fields form a cocycle whose cohomology class will be, by definition, $\rho_0(u)$. p. 4-09

Let $x \in U_{ij}$, and let $\widetilde{x}_i$ and $\widetilde{x}_j$ be its inverse image in $\widetilde{U}_i$ and $\widetilde{U}_j$ (respectively). We have that $\widetilde{x}_j = \widetilde{x}_i + \gamma_{ij}(x)$, where $\gamma_{ij}(x) \in \Gamma$, and

$$w_{ij}(x) = \overline{h}(\widetilde{x}_j) - [\gamma_{ij}(x)]_*(\overline{h}(\widetilde{x}_i)) = -h(\gamma_{ij}(x)) \in \alpha.$$

Now w_{ij} is a vector field on D, and so

$$(w_{ij}) \in \mathrm{Z}^1(V_0,(U_i \times D);\Theta_2),$$

and w_{ij} is of the form $\zeta \otimes \alpha$, where $\zeta \in \mathrm{Z}^1(V_0,(U_i \times D);\mathcal{O})$ is the cocycle defined by $\zeta_{ij}(x) = -\eta(\gamma_{ij}(x))$. This is a cocycle whose cohomology class is (up to a sign) the element of $\mathrm{H}^1(V_0,\mathcal{O})$ that is identified with the class η in $\mathrm{Hom}(\Gamma,\mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E,\mathbb{C})$. QED.

Appendix: Higher obstructions

I. Definition of obstructions

1. The sheaf of germs of vertical automorphisms

Let V_0 be a $\mathbb{C}$-analytic manifold, which we assume to be compact, and B a $\mathbb{C}$-analytic space, and let $b_0 \in B$. We are going to define a sheaf Γ of non-abelian groups on V_0. For every open *p. 4-10*
subset U of V_0, consider the isomorphisms of analytic varieties $\gamma\colon W \to W'$, where W and W' are open subsets of $B \times V_0$ that contain $\{b_0\} \times U$, such that the following conditions are satisfied:

1. $\pi_1\gamma = \pi_1$ is the projection $B \times V_0$ to B;
2. γ is the identity on $\{b_0\} \times U$.

Then $\Gamma(U)$ consists of equivalence classes of these isomorphisms, where we identify γ_1 with γ_2 if they agree on a neighbourhood of $\{b_0\} \times U$.

It is clear that $\Gamma(U)$ is a group under composition of isomorphisms, and that the $\Gamma(U)$ form a sheaf Γ of non-abelian groups.

Proposition 1. *We can identify* $\mathrm{H}^1(V_0,\Gamma)$ *with the set of classes of deformation germs of* V_0 *over* (B,b_0).

Recall that a deformation germ of V_0 over (B,b_0) is a deformation of V_0 over a neighbourhood of b_0 in B, and that two such deformations (B',b_0,V',π',ι') and $(B'',b_0,V'',\pi'',\iota'')$ are locally equivalent if there exists a neighbourhood W' of $(\pi')^{-1}(b_0)$ in V', a neighbourhood W'' of $(\pi'')^{-1}(b_0)$ in V'', and an isomorphism φ from W' to W'' such that the diagram

$$\begin{tikzcd} V_0 \arrow[r, equal] \arrow[d] & V_0 \arrow[d] \\ W' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & W'' \arrow[d, "\pi''"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

commutes. *p. 4-11*

Proof. *(Proof of Proposition 1).* Let (B', b_0, V, π, ι) be a deformation of V_0 over a neighbourhood V' of b_0 in B. Then we can find a cover $\{U_i\}$ of V_0 and a cover $\{W_i\}$ of a neighbourhood of $\iota(V_0)$ in V, along with isomorphisms $\{h_i\}$, where h_i is an isomorphism from a neighbourhood of $\{b_0\} \times U_i$ in $B \times V_0$ to W_i that agrees with ι on $\{b_0\} \times U_i$, and such that $\pi \circ h_i = \pi_1$.

Set $\gamma_{ij} = h_i^{-1} \circ h_j$. We can show that the γ_{ij} define an element of $\Gamma(U_i \cap U_j)$, and that $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$. The γ_{ij} thus form a cocycle $\gamma \in \mathrm{Z}^1(V_0, \{U_i\}; \Gamma)$. Such a cocycle is said to be *associated to the deformation*. It will still be associated to the deformation if pass to a finer cover. Let $(B', b_0, V', \pi', \iota')$ be a deformation that is locally equivalent to the first, and let γ' be a cocycle associated to this deformation. We can suppose, by refining the covers if necessary, that the cocycles γ and γ' are defined with respect to the same cover $\{U_i\}$ of V_0. Let f be an isomorphism from a neighbourhood of $\iota(V_0)$ in V to a neighbourhood of $\iota'(V_0)$ in V'. Set $f_i = (h'_i)^{-1} \circ f \circ h_i$. Then $f_i \in \Gamma(U_i)$, and

$$f_i \circ \gamma_{ij} = \gamma'_{ij} \circ f_j.$$

We thus conclude that the cocycles associated to a deformation form a cohomology class that depends only on the local class of the deformation.

Conversely, suppose we have a locally finite cover $\{U_i\}$ of V_0 and a cocycle $\gamma \in \mathrm{Z}^1(V_0, \{U_i\}; \Gamma)$. Then γ_{ij} can be represented by an isomorphism from an open W_{ij} of $B \times V_0$ to another open W_{ji}, with the two open subsets both containing $\{b_0\} \times U_{ij}$. Pick a refinement $\{U'_i\}$ of the cover $\{U_i\}$, and take some neighbourhood B'' of b_0 in B small enough such that $B'' \times U'_{ij} \subset W_{ij}$ for all (i,j), and such that the equality $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$ holds wherever it is defined in $B'' \times U'_{ijk}$. We thus obtain a deformation V of V_0 on B'' by gluing the $B'' \times U'_i$ via the γ_{ij}.

Finally, we can show that all the above does indeed define a bijection between the set of local classes of deformations of V_0 over (B, b_0) and $\mathrm{H}^1(V_0; \Gamma)$. □

2. Higher obstructions

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For every open subset $U \subset V_0$, the group $\Gamma(U)$ is naturally filtered: denote by $\mathscr{F}_k(U)$ the group of vertical automorphisms that are tangent to the identity up to order $k-1$. Then Γ becomes a filtered sheaf:

$$\Gamma = \mathscr{F}_1 \supset \mathscr{F}_2 \supset \ldots \qquad \text{and } \bigcap \mathscr{F}_k = \{0\}.$$

Set

$$\begin{aligned} \mathscr{Q}_k &= \Gamma/\mathscr{F}_{k+1} \\ \mathscr{G}_k &= \mathscr{F}_k/\mathscr{F})_{k+1} = \mathrm{Ker}(\mathscr{Q}_k \to \mathscr{Q}_{k-1}). \end{aligned}$$

For all k, $\mathscr{G}_k$ is a sheaf of abelian groups, which we will write additively. If $B = \mathbb{C}$ and $b_0 = 0$ (we then speak of *the deformation in one parameter*), for all k, $\mathscr{G}_k$ can be identified with the sheaf Θ of germs of vector fields tangent to V_0. In the general case,

$$\mathscr{G}_k = \mathfrak{m}^k/\mathfrak{m}^{k+1} \otimes \Theta$$

where $\mathfrak{m}$ is the maximal ideal of the point b_0 in B.

Now, if $a \in \mathscr{F}_p$ and $b \in \mathscr{F}_q$, then the commutator $aba^{-1}b^{-1}$ is in $\mathscr{F}_{p+q}$, and this defines a map $\mathscr{G}_p \otimes \mathscr{G}_q \to \mathscr{G}_{p+q}$ which endows $\mathscr{G}_\bullet = \bigoplus \mathscr{G}_k$ with the structure of a sheaf of Lie algebras that is isomorphic to the tensor product of Θ with the graded algebra associated to the maximal ideal $\mathfrak{m}$ of b_0 in B filtered by powers.

The exact sequence of non-abelian groups

$$0 \to \mathscr{G}_{k+1} \to \mathscr{Q}_{k+1} \to \mathscr{Q}_k \to 0$$

in which $\mathscr{G}_{k+1}$ is a subgroup of $\mathscr{Q}_{k+1}$ contained in its centre gives rise [1] to an exact sequence of pointed sets p. 4-13

$$\mathrm{H}^1(V_0;\mathscr{Q}_{k+1}) \to \mathrm{H}^1(V_0;\mathscr{Q}_k) \xrightarrow{\delta_k} \mathrm{H}^2(V_0;\mathscr{G}_{k+1})$$

i.e. for an element $q \in \mathrm{H}^1(V_0,\mathscr{Q}_k)$ to be in the image of $\mathrm{H}^1(V_0;\mathscr{Q}_{k+1})$, it is necessary and sufficient for $\delta_k q = 0$ in $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$. A *necessary* condition for q to be in the image of $\mathrm{H}^1(V_0;\Gamma) \to \mathrm{H}^1(V_0;\mathscr{Q}_k)$ is thus $\delta_k q = 0$ in $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$.

Definition. Let $q \in \mathrm{H}^1(V_0;\mathscr{Q}_i)$, and let $k \geqslant i$. We define an *obstruction of order k of the element q* to be the direct image in $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$ under δ_k of the inverse image of q in $\mathrm{H}^1(V_0;\mathscr{Q}_k)$. It is thus a subset of $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$. The obstruction is said to be *trivial* if the identity element belongs to this subset. Being trivial is a necessary and sufficient condition for q to be in the image of $\mathrm{H}^1(V_0;\mathscr{Q}_{k+1})$, and a necessary condition for q to be in the image of $\mathrm{H}^1(V_0;\Gamma)$.

Warning. If q is not in the image of $\mathrm{H}^1(V_0,\mathscr{Q}_k)$, then its obstruction of order k is empty, and thus non-trivial.

This definition is used most of all in the case of deformations in one parameter ($B = \mathbb{C}$ and $b_0 = 0$), where $\mathscr{G}_{k+1} = \Theta$ for all k, and $\mathscr{Q}_1 = \mathscr{G}_1 = \Theta$. The successive obstructions of an element $a \in \mathrm{H}^1(V_0;\Theta)$ are thus subsets of $\mathrm{H}^2(V_0;\Theta)$, and for a to be a deformation vector, it must be the case that all of its obstructions are trivial. Indeed, the element of $\mathrm{H}^1(V_0;\Theta)$ that corresponds, under the identifications we have made ($\Theta = \mathscr{Q}_1 = \Gamma/\mathscr{F}_2$, and Proposition 1), to a deformation germ is exactly the image under the Spencer–Kodaira map ρ of the canonical basis vector of the tangent space to $\mathbb{C}$ at 0.

II. Calculation of obstructions

1. Relation to the sheaf Ω

From now on, we work in the case of deformations in one parameter, i.e. $B = \mathbb{C}$ and $b_0 = 0$.

Let Ω be the sheaf of universal enveloping algebras of the Lie algebras of the sheaf Θ (i.e. $\Omega(U)$ is the universal enveloping algebra of $\Theta(U)$). p. 4-14

Then Ω contains Θ as a subsheaf, and even as a direct factor (by the Poincaré–Birkhoff–Witt Theorem in characteristic 0). For all k, consider the sheaf of algebras $\Omega_k = \Omega[t]/(t^{k+1})$. For $i \leqslant k$, we have a map of sheaves of sets

$$\exp_i\colon \Theta \to \Omega_k$$

defined by

$$\exp_i(\Theta) = \sum_p \frac{1}{M}\Theta^p t^p$$

Proposition 2. (Campbell–Hausdorff). *We can identify $\mathscr{Q}_k$ with the sheaf of multiplicative subgroups of Ω_k generated by the images of the* $\exp_i$ *for $i \leqslant k$.*

The proof of this proposition will not be given here. We denote by $\Omega_k^\times$ the sheaf of multiplicative subgroups of Ω_k consisting of the elements whose constant terms is 1. The commutative diagram of sheaves of (non-abelian) groups

$$\begin{tikzcd} 0 \arrow[r] & \Theta \arrow[r] \arrow[d] & \mathscr{Q}_{k+1} \arrow[r] \arrow[d] & \mathscr{Q}_k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega \arrow[r] & \Omega_{k+1}^\times \arrow[r] & \Omega_k^\times \arrow[r] & 0 \end{tikzcd}$$

gives rise to a commutative diagram of sets

$$\begin{tikzcd} \mathrm{H}^1(V_0;\mathscr{Q}_k) \arrow[r, "\delta_k"] \arrow[d] & \mathrm{H}^2(V_0;\Theta) \arrow[d] \\ \mathrm{H}^1(V_0;\Omega_k^\times) \arrow[r, "\delta_k"'] & \mathrm{H}^2(V_0;\Omega) \end{tikzcd}$$

in which $\mathrm{H}^2(V_0;\Theta)$ is a vector subspace of $\mathrm{H}^2(V_0;\Omega)$.

2. Calculation of the primary obstruction

Now let $a \in \mathrm{H}^1(V_0;\Theta)$, and let $\alpha = (\alpha_{ij})$ be a cocycle of the class a (the choice of the cocycle α does not matter, since every cocycle that is cohomologous to a deformation cocycle is itself a deformation cocycle). The corresponding multiplicative cocycle in $\Omega_1^\times$ is $(1+\alpha_{ij}t)$. This cocycle can be lifted to $\Omega_i^\times$ as the cochain $(1+\alpha_{ij}t)$, and we have

$$\begin{aligned}(1+\alpha_{ij}t)(1+\alpha_{jk}t) &= 1+(\alpha_{ij}+\alpha_{jk})t+\alpha_{ij}\alpha_{jk}t^2\\ &= (1+\alpha_{ik}t+\alpha_{ij}\alpha_{jk}t^2)\\ &= (1+\alpha_{ik}t)\{1+\alpha_{ij}\alpha_{jk}t^2).\end{aligned}$$

Finally, let

$$\delta_1 a = a \smile a$$

where the cup product is taken in the sheaf of algebras Ω.

Note that, if we denote by $\bar{\smile}$ the cup product taken in the sheaf of algebras opposite to Ω, i.e. defined on the level of cochains by $(\alpha\bar{\smile}\beta)_{ijk} = \beta_{jk}\alpha_{ij}$, we always have that $a\bar{\smile}b = -b \smile a$ in cohomology.

Consequently,

$$[a \smile a] = (a \smile a) - (a\bar{\smile}a) = 2a \smile a$$

and $\delta_1 a = a \smile a = \frac{1}{2}[a \smile a]$. We thus recover, up to a factor of $\frac{1}{2}$, the obstruction defined earlier in this talk.

3. Calculation of the secondary obstruction

Now suppose that $a \smile a = 0$, so that we can find a cochain $\beta = (\beta_{ij})$ such that $\delta\beta + \alpha \smile \alpha = 0$, i.e.

$$\beta_{ik} = \beta_{ij} + \beta_{jk} + \alpha_{ij}\alpha_{jk}.$$

Then $(1+\alpha_{ij}t+\beta ijt^2)$ is a cocycle in $\Omega_2^\times$, and we can choose the cochain β to be a cocycle in $\mathscr{Q}_2$. p. 4-16

This cocycle can be lifted to $\Omega_3^\times$ as the cochain $(1+\alpha_{ij}t+\beta_{ij}t^2)$, and we have that

$$\begin{aligned}&(1+\alpha_{ij}t+\beta_{ij}t^2)(1+\alpha_{jk}t+\beta_{jk}t^2)\\ &= 1+(\alpha_{ij}+\alpha_{jk})t+(\beta_{ij}+\beta_{jk}+\alpha_{ij}\alpha_{jk})t^2+(\alpha_{ij}\beta_{jk}+\beta_{ij}\alpha_{jk})t^3\\ &= (1+\alpha_{ik}t+\beta ikt^2)(1+(\alpha_{ij}\beta_{jk}+\beta_{ij}\alpha_{jk})t^3).\end{aligned}$$

The secondary obstruction of a is thus the cohomology class of the cocycle $(\alpha_{ij}\beta_{jk}+\beta_{ij}\alpha_{jk}) \in \mathrm{Z}^2(V_0;\Omega)$. This class depends on the choice of the cochain β: if we choose some other $\beta' = \beta+\theta$, where $\Theta \in \mathrm{Z}^1(V_0;\Theta)$, then the cocycle is modified by $\alpha \smile \theta + \theta \smile \alpha$, and its class by an element of $[a \smile \mathrm{H}^1(V_0;\Theta)]$. We recover the *Massey triple product* (a,a,a) taken in the algebra Ω, but with a slightly more restrictive indetermination.

We can try to calculate this secondary obstruction without leaving the sheaf Θ, but the calculations are then much more complicated: we must take a cochain $\beta = (\beta_{ij})$ such that $\delta\beta + \frac{1}{2}[a \smile a] = 0$. Then the secondary obstruction of a is the class of the cocycle

$$[\alpha_{ij}, \beta_{jk}] + \frac{1}{6}[[\alpha_{ij}, \alpha_{jk}], \alpha_i j + 2\alpha_{jk}].$$

The calculation done in the sheaf of enveloping algebras Ω can be generalised to obstructions of order r: we are led to determining, by induction, cochains ω_r such that

$$\begin{cases} \omega_1 = \alpha \\ \delta\omega_r + \sum_{p+q=r} \omega_p \smile \omega_q = 0 \\ 1 + \sum_{1 \leqslant p \leqslant r} \omega_p t^p \in \mathrm{C}^1(V_0; \mathscr{Q}_r) \end{cases}$$

4. Using spectral sequences

p. 4-17

Proposition 3. *Let $\varphi\colon V_0 \to X$ be an arbitrary map, which gives rise to a spectral sequence of graded Lie algebras*

$$\mathrm{H}^\bullet(X; \mathbb{R}^\bullet\varphi\Theta) \Rightarrow \mathrm{H}^\bullet(V_0;\Theta).$$

Let

$$a \in \mathrm{H}^1(X; \varphi_*\Theta) \subset \mathrm{H}^1(V_0;\Theta).$$

If the element

$$-\frac{1}{2}[a \smile a] \in \mathrm{H}^2(X;\varphi_*\Theta) = E_2^{2,0}$$

is non-zero, but is the image under the differential d_2 of the spectral sequence of an element $b \in E_2^{0,1}$, then the image of the secondary obstruction of a in $E_\infty^{1,1}$ consists of the elements of the form $[a,b]$. In particular, if, for all b such that $\mathrm{d}_2 b = -\frac{1}{2}[a,a]$, we have that $[a,b] \neq 0$, then the secondary obstruction is non-trivial.

Warning. However, if $[a,b] = 0$ in $E^{1,1}$, then we can only say that the secondary obstruction comes from $E_\infty^{2,0}$, and if this group is non-zero, then we cannot conclude anything.

Proof. Let α be a cocycle on V_0 representing the class a. The element $b \in E_2^{0,1}$ can be represented by a cochain

$$\beta = (\beta_{ij}) \in \mathrm{C}^1(V_0;\Theta)$$

such that

p. 4-16

$$\delta\beta + \frac{1}{2}[a \smile a] = 0.$$

We thus obtain a cochain

$$\beta' \in \mathrm{C}^1(V_0;\Omega)$$

such that

$$1 + \alpha t + \beta' t^2 \in \mathrm{C}^1(V_0;\mathscr{Q}_2)$$

by setting $\beta'_{ij} = \beta_{ij} + \frac{1}{2}\alpha_{ij}^2$; this cochain satisfies $\delta\beta' + \alpha \smile \alpha = 0$. But this new cochain represents, in the $E_2^{0,1}$ term of the spectral sequence of the sheaf Ω, the same element b as the cochain β, since it differs from it by a cochain that comes from X. The secondary obstruction is thus the class of the cocycle $\alpha \smile \beta' + \beta' \smile \alpha$, which represents in the $E^{1,1}$ term of the spectral sequence the element $[a,b]$. $\square$

This proposition allows us to construct non-trivial examples of secondary obstructions. Consider the group N of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x,y,z \in \mathbb{C}$, and let $Y = N/\Gamma$, where Γ is the subgroup of N consisting of elements where $x,y,z \in \mathbb{Z}+i\mathbb{Z}$. Then Y is fibred over a complex torus of dimension two $T^2 \cong \mathbb{C}^2/\mathbb{Z}^4$. We find non-trivial secondary obstruction elements in $\mathrm{H}^1(V_0;\Theta)$, where V_0 is the product of Y with a projective line D. (We use the spectral sequence obtained by projecting onto $T^2 \times D$). This variety has a "versal" deformation whose Zariski tangent space of the base B can be identified via the Spencer–Kodaira map ρ with $\mathrm{H}^1(V_0;\Theta)$. Further, B has, at its base point b_0, a conic singularity of degree 3, whose equation is given by the secondary obstruction. *p. 4-19*

I do not know of any examples of non-trivial secondary obstructions on varieties V_0 that satisfy $\mathrm{H}^0(V_0;\Theta) = 0$, but some very likely exist.

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