

Regular deformations

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I. The map $\tilde{\rho}$

All throughout this talk, B is a $\mathcal{C}^\infty$ manifold (resp. $\mathbb{R}$ -analytic, resp. $\mathbb{C}$ -analytic); $\pi: V \rightarrow B$ denotes a proper mixed manifold; b_0 is a point of B ; and $V_0 = \pi^{-1}(b_0)$ is thus a compact $\mathbb{C}$ -analytic manifold.

Let $\tilde{\Theta}$ (resp. $\tilde{\Pi}$) be the sheaf of germs of vertical holomorphic (resp. locally projectable holomorphic) vector fields on V . The quotient sheaf $\tilde{\Lambda} = \tilde{\Pi}/\tilde{\Theta}$ is exactly the inverse image under π of the sheaf $\tilde{T}$ of germs of $\mathcal{C}^\infty$ fields (resp. . . .) of tangent vectors on B .

For every open subset U of B , set $V_U = \pi^{-1}(U)$. The exact sequence

$$0 \rightarrow \tilde{\Theta} \rightarrow \tilde{\Pi} \rightarrow \tilde{\Lambda} \rightarrow 0$$

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of sheaves on V_U gives rise to a homomorphism

$$\tilde{\rho}_U: H^0(U; \tilde{T}) \xrightarrow{\pi_*} H^0(V_U; \tilde{\Lambda}) \xrightarrow{\delta} H^1(V_U; \tilde{\Theta}).$$

Let $R^1\pi_*\tilde{\Theta}$ be the sheaf on B defined by the presheaf $U \mapsto H^1(V_U; \tilde{\Theta})$. Then $\tilde{\rho}$ becomes a homomorphism of sheaves on B :

$$\tilde{\rho}: \tilde{T} \rightarrow R^1\pi_*\tilde{\Theta}.$$

In particular, we have a homomorphism

$$\tilde{\rho}_0: \tilde{T}_0 \rightarrow R^1\pi_*\tilde{\Theta} = H^1(V_0; \tilde{\Theta})$$

where $\tilde{T}_0$ is the vector space of germs at b_0 of fields of tangent vectors to B . Finally, we have | p. 3-02
a commutative diagram

$$\begin{array}{ccc} \tilde{T}_0 & \xrightarrow{\tilde{\rho}_0} & H^1(V_0; \tilde{\Theta}) \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ T_0 & \xrightarrow{\rho_0} & H^1(V_0; \Theta_0) \end{array}$$

where ρ_0 is the Spencer–Kodaira map [2].

Theorem 1. *For the proper mixed manifold $\pi: V \rightarrow B$ to be locally trivial in a neighbourhood of the point $b_0 \in B$, it is necessary and sufficient for the map $\tilde{\rho}_0: \tilde{T}_0 \rightarrow H^1(V_0; \tilde{\Theta})$ to be zero.*

Proof. —

- a. (Necessity). If $\pi: V \rightarrow B$ is locally trivial at b_0 , then, for every open subset U of B over which V is trivial, we have $\tilde{\Pi} = \tilde{\Lambda} \oplus \tilde{\Theta}$ on V_U , and so $\delta: H^0(V_U; \tilde{\Lambda}) \rightarrow H^0(V_U; \tilde{\Theta})$ is zero.
- b. (Sufficiency). Let $(\eta_1, \dots, \eta_p)$ be $\mathcal{C}^\infty$ vector fields (resp. ...) on a neighbourhood of b_0 in B , such that $(\eta_1(b_0), \dots, \eta_p(b_0))$ forms a basis of the tangent space T_0 to B at b_0 . It then follows from the hypothesis that the map

$$H^0(V_0; \tilde{\Pi}) \rightarrow H^0(V_0; \tilde{\Lambda})$$

is surjective. So let $(\xi_1, \dots, \xi_p)$ be projectable holomorphic vector fields on a neighbourhood of V_0 in V , that project to $(\eta_1, \dots, \eta_p)$. Let f be the map defined on a neighbourhood of $\{0\} \times V_0$ in $\mathbb{R}^p \times V_0$ (resp. $\mathbb{C}^p \times V_0$) by

$$f(t_1, \dots, t_p, y) = e^{\xi_1}(t_1, e^{\xi_2}(\dots, e^{\xi_p}(t_p, y) \dots)).$$

It follows from the proposition stated in [1, Section III.2] that f induces an isomorphism of mixed manifolds from $U \times V_0$ to $\pi^{-1}(f_1(U))$ over f_1 , where U is a sufficiently small cubical neighbourhood of 0 in $\mathbb{R}^p$, and f_1 is the map from U to B defined by

$$f_1(t_1, \dots, t_p) = e^{\eta_1}(t_1, e^{\eta_2}(\dots, e^{\eta_p}(t_p, b_0) \dots)),$$

which proves the theorem.

□ | p. 3-03

II. The regular case

For all $b \in B$, set $V_b = \pi^{-1}(b)$. Consider the family $\{H^1(V_b; \Theta_b)\}_{b \in B}$ of finite-dimensional $\mathbb{C}$ -vector spaces, and, for all $b \in B$, the map

$$\varepsilon_b: H^1(V_b; \tilde{\Theta}) \rightarrow H^1(V_b; \Theta_b).$$

For every open subset $U \subset B$, we have a map

$$\tilde{\varepsilon}_U: H^1(V_U; \tilde{\Theta}) \rightarrow \prod_{b \in U} H^1(V_b; \Theta_b)$$

that defines, by varying U , a homomorphism from the sheaf $R^1\pi_*\tilde{\Theta}$ to the sheaf Φ on B defined by $\Phi(U) = \prod_{b \in U} H^1(V_b; \Theta_b)$.

Definition.

We say that the proper mixed manifold $\pi: V \rightarrow B$ is *regular* if

1. the dimension of $H^1(V_b; \Theta_b)$ does not depend on the point $b \in B$; and
2. we can endow $E = \bigcup_{b \in B} H^1(V_b; \Theta_b)$ with the structure of a $\mathcal{C}^\infty$ vector bundle (resp. ...) such that $\tilde{\varepsilon}$ is an isomorphism from the sheaf $R^1\pi_*\tilde{\Theta}$ to the sheaf of germs of $\mathcal{C}^\infty$ sections (resp. ...) of the bundle E .

In fact, Kodaira and Spencer have shown [2] that, by identifying the H^1 spaces with spaces of harmonic forms, condition (2) is a consequence of condition (1).

Then **Theorem 1** has the following corollary:

Proposition 1. *For the proper mixed manifold $\pi: V \rightarrow B$ to be locally trivial, it is necessary and sufficient for it to be regular and, for all $b \in B$, for the Spencer–Kodaira map*

$$\rho_b: T_b \rightarrow H^1(V_b; \Theta_b)$$

to be zero.

Indeed, since $\tilde{\varepsilon}$ is injective, this condition implies that the map

$$\tilde{\rho}_b: \tilde{T}_b \rightarrow H^1(V_b; \tilde{\Theta})$$

is zero for all b .

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At the end of this talk, we will construct a counter-example which shows that it is necessary to assume that the mixed manifold is regular.

III. An example of non-regular deformation: Hopf manifolds

1. Hopf manifolds

Let $n \geq 2$ be an integer, and let b be an $(n \times n)$ matrix with coefficients in $\mathbb{C}$, whose eigenvalues are all of modulus > 1 . The free group $L(b)$ generated by b acts freely on $\tilde{V} = \mathbb{C}^n \setminus \{0\}$, and the quotient space $\tilde{V}/L(b)$, which we call the *Hopf manifold defined by b* , is a compact $\mathbb{C}$ -analytic manifold that is homeomorphic to $S^{2n-1} \times S^1$.

Note that V_b and $V_{b'}$ are isomorphic if and only if there exists some a such that $b' = aba^{-1}$ or $b' = ab^{-1}a^{-1}$ (cf. [Appendix](#)).

Let Θ be the sheaf of germs of holomorphic fields of tangent vectors on V_b .

Proposition 2. *We can identify $H^0(V_b; \Theta)$ with the vector space of matrices that commute with b , and $H^1(V_b; \Theta)$ has the same dimension as this vector space.*

Proof. If X is a vector field on an open subset $U \subset \tilde{V}$, then $b_*(X)$ is the vector field on the open subset $b(U)$ given by transporting via b , i.e. $b_*X(u) = bX(b^{-1}u)$. Let $\mathcal{U} = \{U_i\}$ be a cover of V by simply connected Stein open subsets; for all i , set $\tilde{U}_i = \chi^{-1}\{U_i\}$, where χ is the canonical map from $\tilde{V}$ to V_b . The cover $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ of $\tilde{V}$ consists of Stein open subsets that are invariant under b (not necessarily connected, but this doesn't matter). Then b_* defines a map, again denoted by b_* , from the group of cochains $C^\bullet(\tilde{V}, \tilde{\mathcal{U}}; \Theta)$ to itself.

Lemma 1. *We have the exact sequence*

$$0 \rightarrow C^\bullet(V_b, \mathcal{U}; \Theta) \xrightarrow{\chi^*} C^\bullet(\tilde{V}, \tilde{\mathcal{U}}; \Theta) \xrightarrow{1-b_*} C^\bullet(\tilde{V}, \tilde{\mathcal{U}}; \Theta) \rightarrow 0.$$

Proof. The only thing that we need to verify is that the map $1 - b_*$ is surjective. For all $(i_0, \dots, i_q)$, let $U'_{i_0, \dots, i_q}$ be an open subset of $\tilde{V}$ such that

$$\chi: U'_{i_0, \dots, i_q} \rightarrow U_{i_0, \dots, i_q}$$

is a homeomorphism. The $\tilde{U}_{i_0, \dots, i_q}$ is a disjoint union of the $b_*^p U'_{i_0, \dots, i_q}$, where $p \in \mathbb{Z}$, and every $\gamma \in C^q(\tilde{V}, \tilde{\mathcal{U}}; \Theta)$ can be written in the form $\gamma = \gamma_1 - \gamma_2$, with $\gamma_1 = 0$ on $b^p(U'_{i_0, \dots, i_q})$ for $p < 0$, and $\gamma_2 = 0$ for $p \geq 0$. Set

$$\beta = \sum_{p \geq 0} b_*^p \gamma_1 + \sum_{p < 0} b_*^p \gamma_2$$

(which is a locally finite sum). Then $\beta - b_*\beta = \gamma$, whence [Lemma 1](#). □

Now, to finish the proof of [Proposition 2](#). From [Lemma 1](#), we have the following exact sequence:

$$0 \rightarrow H^0(V_b; \Theta) \xrightarrow{\chi^*} H^0(\tilde{V}; \Theta) \xrightarrow{1-b_*} H^0(\tilde{V}; \Theta) \xrightarrow{\delta_*} H^1(V_b; \Theta) \xrightarrow{\chi^*} H^1(\tilde{V}; \Theta) \xrightarrow{1-b_*} H^1(\tilde{V}; \Theta).$$

We can show that

$$\chi^*: H^1(V_b; \Theta) \rightarrow H^1(\tilde{V}; \Theta)$$

is zero: if $n > 2$, it is evident, since $H^1(\tilde{V}; \Theta) = 0$; if $n = 2$, then a direct calculation on the cochains of a cover of $\tilde{V}$ by two Stein open subsets shows that

$$1 - b_*: H^1(\tilde{V}; \Theta) \rightarrow H^1(\tilde{V}; \Theta)$$

is bijective.

Now $H^0(\tilde{V}; \Theta)$ is the space of holomorphic vector fields on $\tilde{V}$, but such a field extends to a holomorphic vector field on $\mathbb{C}^n$, and $H^0(\tilde{V}, \Theta) = L \oplus M$, where L is the space of fields of linear vectors, and M is the space of fields of second-order vectors at 0. The subspaces L and M are invariant under b_* , and $1 - b_*: M \rightarrow M$ is an isomorphism. Then [Proposition 2](#) follows from remarking that, if an element of L is represented by a matrix a , then $b_*a = bab^{-1}$. □

2. Mixed manifolds whose fibres are Hopf manifolds

Let B be the set of all $(n \times n)$ matrices with coefficients in $\mathbb{C}$ with eigenvalues all of modulus > 1 . This is an open subset of $\mathbb{C}^{n^2}$. Let α be the transformation from $B \times \tilde{V}$ to itself defined by $\alpha(b, x) = (b, b(x))$. The free group $L(\alpha)$ generated by α acts linearly on $B \times \tilde{V}$, and the quotient $V = B \times \tilde{V}/L(\alpha)$ is a $\mathbb{C}$ -analytic manifold. By endowing it with the projection $\pi: V \rightarrow B$ induced by the projection $\pi_1: B \times \tilde{V} \rightarrow B$ after passing to the quotient, we obtain a $\mathbb{C}$ -analytic mixed manifold that is proper, but not regular. Indeed, condition 1 of the definition of regular mixed manifolds is not satisfied: for example, for $n = 2$, the dimension of $H^1(V_b; \Theta)$ is 4 if b is a scalar matrix, but 2 in all other cases. | p. 3-06

Note that the dimension of $H^1(V_b; \Theta_b)$ is an upper semi-continuous function of b , and that the set of b such that $\dim H^1(V_b; \Theta_b) \geq k$ is a closed analytic subspace of B . This is a general result, that we hope to be able to prove in a later talk of this seminar.

3. Calculation of ρ

We have $T_b = \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) = L \subset H^0(\tilde{V}; \Theta)$, and we defined, to prove [Proposition 2](#), a surjective map $\delta_*: L \rightarrow H^1(V_b; \Theta)$.

Proposition 3. *The Spencer–Kodaira map ρ is given, for the mixed manifold studied in this section, by*

$$\rho(a) = \delta_*(ab^{-1}).$$

In particular, it is surjective, and its kernel is the space of matrices of the form $[\ell, b]$ for $\ell \in L$.

Proof. Let $a \in T_b = L$. Let $\{U_i\}$ be a cover of V_b by simply connected Stein open subsets, and, for each i , let U'_i be a connected component of $\tilde{U}_i$.

Let η'_i be the projectable holomorphic field on U'_i defined by $\eta'_i(x) = (a, 0)$; let $\tilde{\eta}_i$ be the projectable holomorphic field on $\tilde{U}_i$ defined by $\tilde{\eta}_i = \alpha_*^k \eta'_i$ on $b^k(U'_i)$; and let η_i be the projectable holomorphic field on U_i corresponding to $\tilde{\eta}_i$. By definition, $\rho(a)$ is the cohomology class of the cochain $\{\theta_{ij}\}$, where $\theta_{ij} = \eta_j - \eta_i$ is a vertical holomorphic field on U_{ij} . | p. 3-07

Set $\tilde{\eta}_i(x) = (a, \beta_i(x))$. Then $\beta \in C^0(\tilde{V}; \Theta)$, and we have $(1 - b_*)\beta = ab^{-1} \in L \subset H^0(\tilde{V}; \Theta)$. Indeed, $\alpha_*\eta = \eta$, $\alpha_*\eta_i(b_{-1}x) = \eta_i(x)$, and

$$\alpha_*(a, \beta(b^{-1}x)) = (a, \beta(x)),$$

whence

$$ab^{-1}x + b \cdot \beta(b^{-1}x) = \beta(x).$$

We thus deduce that $\theta = \delta_*(ab^{-1})$, which proves [Proposition 3](#). □

4. A counter-example

Take $n = 2$, and $\sigma \in \mathbb{C}$ such that $|\sigma| > 1$. Let $B' \subset B$ be the set of matrices of the form

$$\begin{pmatrix} \sigma & t \\ 0 & \sigma \end{pmatrix}$$

where $t \in \mathbb{C}$, and let $V' = \pi^{-1}(B')$ be the mixed manifold induced by V over V' ; now B' is a line, and its tangent space T'_b at b is generated, for all b , by $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It follows from [Proposition](#)

3 that the Spencer–Kodaira map

$$\rho': T_b(B') \rightarrow H^1(V_b; \Theta)$$

is zero if and only if

$$b \neq b_0 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

since, if $b \neq b_0$, then $a = [\ell, b]$, where $\ell = \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}$; and if $b = b_0$, then ρ' is injective.

We can also see that V' is trivial on $B' \setminus \{b_0\}$.

Let $\varphi: \mathbb{C} \rightarrow B' \subset B$ be the map defined by

$$\varphi(t) = \begin{pmatrix} \sigma & t^2 \\ 0 & \sigma \end{pmatrix}$$

and let V^φ be the mixed manifold given by the inverse image of V under φ . The Spencer–Kodaira map ρ'_t from $\mathbb{C}$ to $H^1(V_{\varphi(t)}; \Theta)$ is the composition

| p. 3-08

$$\rho'_{\varphi(t)} \circ D\varphi: \mathbb{C} \rightarrow T'_{\varphi(t)} \rightarrow H^1(V_{\varphi(t)}; \Theta),$$

and this is zero for all t , since, if $t \neq 0$, then $\rho'_{\varphi(t)}$ is zero; and, if $t = 0$, then $D\varphi$ is zero.

However, the mixed manifold V^φ is not locally trivial, since V_0^φ is not isomorphic to V_t^φ for $t \neq 0$.

5. Question (K. Srinivasacharyulu)

We know that the Hopf manifolds are non-Kähler, and thus non-algebraic. For $n = 2$, the manifold V_b admits non-constant meromorphic functions if and only if b can be diagonalised with eigenvalues σ_1 and σ_2 satisfying $\sigma_1^p = \sigma_2^q$ for some integers p and q (and there is then the function $x_1^p x_2^{-q}$). The set of b satisfying this property is neither open nor closed, but it is a countable union of closed analytic subspaces. An analogous phenomenon arises for deformations of complex tori. Is this result general?

Appendix

With the notation of §III.1, let $f: V_b \rightarrow V_{b'}$ be an isomorphism of $\mathbb{C}$ -analytic manifolds. This lifts to an isomorphism of universal coverings

$$\tilde{f}: \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}^n \setminus \{0\}.$$

By Hartog, $\tilde{f}$ extends to an isomorphism $g: \mathbb{C}^n \rightarrow \mathbb{C}^n$. We necessarily have

$$g(bz) = (b')^k g(z) \tag{*}$$

where $z \in \mathbb{C}^n$, and k is an integer; the same property, applied to the inverse map of g , shows that $k = \pm 1$. Let a be the linear map that is tangent to g at the origin; the identity (*) then gives

$$\begin{aligned} ab &= (b')^k a \\ k &= \pm 1 \end{aligned}$$

whence

$$b' = aba^{-1} \quad \text{or} \quad b' = ab^{-1}a^{-1}.$$

Bibliography

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