

Mixed manifolds and mixed spaces

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I. Category of models

Let B be a topological space. We define the category $\mathcal{S}_B^n$ in the following manner: the objects of $\mathcal{S}_B^n$ are the open subsets of $B \times \mathbb{C}^n$, and a morphism $f: U \rightarrow U'$ from an open subset $U \subset B \times \mathbb{C}^n$ to an open subset $U' \subset B \times \mathbb{C}^n$ is a continuous map $f: U \rightarrow U'$ satisfying the following two conditions:

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ B & \xlongequal{\quad} & B \end{array}$$

commutes, where π_1 denotes the projection of $B \times \mathbb{C}^n$ to B ; and

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2. for all $x \in B$, the map $f_x: U_x \rightarrow U'_x$ is holomorphic, where

$$U_x = \{z \in \mathbb{C}^n \mid (x, z) \in U\}$$

(and similarly for U').

If B is endowed with the structure of a $\mathcal{C}^\infty$ manifold (resp. an $\mathbb{R}$ -analytic manifold, resp. $\mathbb{C}$ -analytic manifold), then we obtain a category $\mathcal{C}^\infty \mathcal{S}_B$ (resp. $\mathbb{R} \mathcal{S}_B$, resp. $\mathbb{C} \mathcal{S}_B$) by requiring the morphisms to be $\mathcal{C}^\infty$ (resp. $\mathbb{R}$ -analytic, resp. $\mathbb{C}$ -analytic).

More generally, if $f_1: B \rightarrow B'$ is a continuous map from one topological space to another, then a *morphism of $\mathcal{S}_{f_1}$* is a continuous map f from an object U of $\mathcal{S}_B$ to an object U' of $\mathcal{S}_{B'}$ such that

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ B & \xrightarrow{f_1} & B' \end{array}$$

commutes; and

2. $f_x: U_x \rightarrow U'_{f_1(x)}$ is holomorphic for all $x \in B$.

If f_1 is a $\mathcal{C}^\infty$ map from one $\mathcal{C}^\infty$ manifold to another, then f will be a morphism of $\mathcal{C}^\infty \mathcal{S}_{f_1}$ if, further, it is a $\mathcal{C}^\infty$ map (resp. ...). We thus obtain, for every category of topological spaces, a fibred category $\mathcal{S}^n$ (resp. $\mathcal{C}^\infty \mathcal{S}^n$, resp. ...).

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II. The definition of mixed spaces and mixed varieties

1. First definition

Let B and V be separated spaces, and let $\pi: V \rightarrow B$ be a continuous map. The structure of a *mixed space* over B is defined on V by a system of charts $\varphi_i: U_i \rightarrow V$, where the (U_i) are objects of $\mathcal{S}_B^n$; for each i , φ_i is a homeomorphism from U_i to an open subset of V such that the diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_i} & V \\ \pi_1 \downarrow & & \downarrow \pi \\ B & \xlongequal{\quad} & B \end{array}$$

commutes; finally, for all i and all j , the “change of chart” $\varphi_j^{-1} \circ \varphi_i$ is an isomorphism of $\mathcal{S}_B$ from an open subset of U_i to an open subset of U_j .

The structure thus defined is that of a $(\mathcal{C}^0, \mathbb{C})$ -mixed space. If B is a $\mathbb{C}$ -analytic space, and if the change of chart maps are all $\mathbb{C}$ -analytic, then we have a $\mathbb{C}$ -analytic mixed space. In this case, V itself is a $\mathbb{C}$ -analytic space, and the fibres $V_x = \pi^{-1}(x)$ are $\mathbb{C}$ -analytic sub-manifolds.

If B is a $\mathcal{C}^\infty$ manifold (resp. $\mathbb{R}$ -analytic, resp. $\mathbb{C}$ -analytic), and if the change of chart maps are all $\mathcal{C}^\infty$ (resp. ...), then we have a $(\mathcal{C}^\infty, \mathbb{C})$ -mixed manifold (resp. $(\mathbb{R}, \mathbb{C})$, resp. $(\mathbb{C}, \mathbb{C})$). In this case, V itself is a manifold. Note that the notion of a $(\mathbb{C}, \mathbb{C})$ -mixed manifold, or a $\mathbb{C}$ -analytic mixed manifold, reduces to simply having a $\mathbb{C}$ -analytic manifold V endowed with a

projection $\pi: V \rightarrow B$ onto another $\mathbb{C}$ -analytic manifold such that π is of maximal rank at every point.¹

Let $\pi: V \rightarrow B$ and $\pi': V' \rightarrow B'$ be mixed spaces, and let $f_1: B \rightarrow B'$ be a continuous (resp. ...) map. Then a *morphism from V to V' over f_1* is a continuous map $f: V \rightarrow V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f_1} & B' \end{array}$$

commutes, and such that, for any charts $\varphi_i: U_i \rightarrow V$ and $\varphi'_j: U'_j \rightarrow V'$, the map $\varphi'^{-1}_j \circ f \circ \varphi_i$ is a morphism of $\mathcal{S}_{f_1}$ (resp. ...) from an open subset of U_i to U_j .

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2. An equivalent definition

We now give another way of defining mixed spaces, equivalent to the above.

Given separated spaces B and V , along with a continuous map $\pi: V \rightarrow B$, the structure of a *pre-mixed space* consists of the structure of a $\mathbb{C}$ -analytic manifold on each fibre $V_x = \pi^{-1}(x)$. Given pre-mixed spaces $\pi: V \rightarrow B$ and $\pi': V' \rightarrow B'$, along with a continuous map $f_1: B \rightarrow B'$, a *morphism of pre-mixed spaces over f_1* is a continuous map $f: V \rightarrow V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f_1} & B' \end{array}$$

commutes and induces a $\mathbb{C}$ -analytic map on each fibre.

A *mixed space* is a pre-mixed space $\pi: V \rightarrow B$ such that every point $y \in V$ admits a neighbourhood W in V that is isomorphic as a pre-mixed space to an open subset of $B \times \mathbb{C}^n$, via an isomorphism over the identity. The morphisms of mixed spaces are the same: mixed spaces form a *full subcategory*.

3. Deformations

A mixed space $\pi: V \rightarrow B$ is said to be *proper* if B is locally compact and the map π is proper (i.e. the inverse image of any compact subset is compact). If it is a mixed manifold, then we can show that it is a fibred manifold that is locally trivial with respect to the underlying $\mathcal{C}^\infty$ structure, but the previous talk shows that, in general, any two fibres are not isomorphic as $\mathbb{C}$ -analytic manifolds.

Definition. Let V_0 be a compact $\mathbb{C}$ -analytic manifold, B a locally compact space, and $b_0 \in B$. Then a *$\mathbb{C}$ -analytic deformation of V_0 over (B, b_0)* consists of a proper $\mathbb{C}$ -analytic mixed space $\pi: V \rightarrow B$ along with an isomorphism of $\mathbb{C}$ -analytic manifolds $i: V_0 \rightarrow \pi^{-1}(b_0)$.

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The goal of this seminar is the study, at least local, and an attempt at a classification of, $\mathbb{C}$ -analytic deformations of a given compact $\mathbb{C}$ -analytic manifold V_0 .

¹[Trans.] The more common modern nomenclature is to simply call such an object a family of complex manifolds.

Definition. Let V_0 be a compact $\mathbb{C}$ -analytic manifold. A $\mathbb{C}$ -analytic deformation $(\pi: V \rightarrow B, i: V_0 \rightarrow V)$ of V_0 is said to be *locally complete* if, for any other deformation $(\pi': V' \rightarrow B', i': V_0 \rightarrow V')$ of V_0 , there exists a neighbourhood B'_1 of b'_0 in B' , an analytic map $f_1: B'_1 \rightarrow B$ with $f_1(b'_0) = b_0$, and a morphism of $\mathbb{C}$ -analytic mixed spaces $f: \pi'^{-1}(B'_1) \rightarrow V$ over f_1 such that $f \circ i' = i$. The deformation is said to be *locally universal* if furthermore the germ of f_1 at b'_0 is determined uniquely by this condition.

It seems that every compact $\mathbb{C}$ -analytic manifold V_0 admits a locally complete $\mathbb{C}$ -analytic deformation, and a locally universal one if the group of automorphisms of V_0 is discrete.

III. Vector fields

1. Study on models

Let B be a space, U an object of $\mathcal{S}_B$ (i.e. an open subset of $B \times \mathbb{C}^n$), b_0 a point of B , and set $U_0 = \pi^{-1}(b_0)$.

A holomorphic field of tangent vectors on U_0 (i.e. a holomorphic map from U_0 to $\mathbb{C}^n$) is said to be a *vertical holomorphic field* on U_0 . A *vertical holomorphic field on U* is a continuous (resp. ...) map $\theta: U \rightarrow \mathbb{C}^n$ that induces a vertical holomorphic field on each fibre U_x . If $f: U \rightarrow U'$ is an isomorphism in $\mathcal{S}_B$, then the *transport $f_*\theta$ of θ by f* is defined by

$$f_*\theta(f(x, z)) = D_2f_{x,z} \cdot \theta(x, z)$$

where $D_2f_{x,z}$ is the linear map from $\mathbb{C}^n$ to itself that is tangent to f_x at the point $z \in U_x$. This is again a vertical holomorphic field, since it follows from a Cauchy integral that the matrix $Df_{x,z}$ depends continuously on the pair (x, z) .

Now suppose that B is a $\mathcal{C}^\infty$ manifold, just for simplicity, and let T_0 be the tangent space to B at b_0 . A field of tangent vectors to U defined on U_0 , i.e. a map $\omega: U_0 \rightarrow T_0 \times \mathbb{C}^n$, is said to be a *projectable holomorphic field* if $\omega(b_0, z) = (t_0, \theta(z))$ (where $t_0 \in T_0$ is a vector that does not depend on z , called the *projection* of the field ω) and $\theta(z)$ is a holomorphic vector field. If B is a $\mathbb{C}$ -analytic space, possibly with a singularity at b_0 , then we give the same definition, but with T_0 then being the *Zariski* tangent space to B at b_0 , i.e. the dual of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the ideal of germs at b_0 of holomorphic functions on B that vanish at b_0 .

If $f: U \rightarrow U'$ is an isomorphism of $\mathcal{C}^\infty\mathcal{S}_B$ (resp. ...), then the transport $f_*\omega$ is defined by

$$f_*\omega(f(b_0, z)) = Df_{b_0,z}\omega(b_0, z)$$

where $Df_{b_0,z}: T_0 \times \mathbb{C}^n \rightarrow T_0 \times \mathbb{C}^n$ is now the linear map that is tangent to f at the point (b_0, z) . This is a projectable holomorphic field. Indeed, the matrix $Df_{b_0,z}$ can be written as

$$\begin{pmatrix} I & 0 \\ D_1f & D_2f \end{pmatrix}$$

and

$$\begin{aligned} D_1f &: T \rightarrow \mathbb{C}^n \\ D_2f &: \mathbb{C}^n \rightarrow \mathbb{C}^n \end{aligned}$$

both depend holomorphically on z (for D_1f , this follows from the fact that f_x is holomorphic for every x). By setting $f_*\omega(b_0, z') = (t_0, \theta'(z'))$, we have

$$\begin{aligned} \theta'(z') &= D_1f_{b_0,z}(t_0) + D_2f_{b_0,z}(\omega(z)) \\ \text{if } z' &= f_{b_0}(z) \end{aligned}$$

which shows that $f_*\omega$ is indeed a projectable holomorphic field.

A *projectable holomorphic field* on U is a $\mathcal{C}^\infty$ field of vectors tangent to U that induces a projectable holomorphic field on each fibre.

2. Vector fields on a mixed manifold

Let $\pi: V \rightarrow B$ be a $(\mathcal{C}^\infty, \mathbb{C})$ -mixed manifold (resp. $\dots$, resp. a $\mathbb{C}$ -analytic mixed space). By transporting along the charts, we define the notions of

- vertical holomorphic fields on an open subset of a fibre;
- vertical holomorphic fields on a open subset of V ;
- projectable holomorphic fields on an open subset of a fibre; and
- projectable holomorphic fields on an open subset of V .

Let ξ be a $\mathcal{C}^\infty$ vector field (resp. $\dots$) on V . By integrating ξ , we obtain a $\mathcal{C}^\infty$ map, denoted by e^ξ , from an open subset $W \subset \mathbb{R} \times V$ containing $\{0\} \times V$ (resp. $\mathbb{C}$ -analytic map from an open subset $W \subset \mathbb{C} \times V$) to V , characterised by

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1. $e^\xi(t_1 + t_2, y) = e^\xi(t_1, e^\xi(t_2, y))$, with the left-hand side being defined whenever the right-hand side is; and
2. $\frac{\partial}{\partial t} e^\xi(t, y)|_{0, y} = \xi(y)$.

Note that W is a mixed manifold over $\mathbb{R} \times B$ (resp. a mixed space over $\mathbb{C} \times B$).

Proposition. For $e^\xi: W \rightarrow V$ to be a morphism of mixed spaces over the projection $\mathbb{R} \times B \rightarrow B$, it is necessary and sufficient for ξ to be a vertical holomorphic field. For $e^\xi: W \rightarrow V$ to be a morphism of mixed spaces over a map from an open subset of $\mathbb{R} \times B$ containing $\{0\} \times B$ to B , it is necessary and sufficient for ξ to be a projectable holomorphic field.

The proof is left to the reader.

IV. The Spencer–Kodaira map

Let $\pi: V \rightarrow B$ be a mixed manifold (resp. a $\mathbb{C}$ -analytic mixed space), $b \in B$, and $V_0 = \pi^{-1}(b)$. Let T_0 be the tangent space to B at b_0 (resp. the Zariski tangent space). We introduce the following sheaves on V_0 :

- Θ_0 : the sheaf of germs of vertical holomorphic fields on V_0 ;
- Π_0 : the sheaf of germs of locally projectable holomorphic fields on V_0 ; and
- Λ_0 : the sheaf π^*T_0 , i.e. the sheaf of germs of locally constant maps from V_0 to T_0 .

We have an exact sequence of sheaves on V_0

$$0 \rightarrow \Theta_0 \rightarrow \Pi_0 \rightarrow \Lambda_0 \rightarrow 0$$

that gives rise to the long exact sequence in cohomology

$$\dots \rightarrow H^0(V_0; \Pi_0) \rightarrow H^0(V_0; \Lambda_0) \xrightarrow{\delta} H^1(V_0; \Theta_0) \rightarrow \dots$$

We also have a canonical map

$$\iota: T_0 \rightarrow H^0(V_0; \Lambda_0)$$

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that is injective if V_0 is non-empty, and surjective if V_0 is connected.

Definition. The *Spencer–Kodaira map* is the composition

$$\rho_0 = \delta \circ \iota: T_0 \rightarrow H^1(V_0; \Theta_0).$$

This map is an essential tool in the local study of deformations of $\mathbb{C}$ -analytic varieties. Note that Θ_0 is exactly the sheaf of germs of holomorphic fields of tangent vectors to V_0 , and thus depends only on V_0 , while T_0 depends only on the base. Also, Θ_0 is a coherent analytic sheaf on V_0 , and, if V_0 is compact, then $H^1(V_0; \Theta_0)$ is a finite-dimensional vector space over $\mathbb{C}$ [1]. We thus see that, in this case (which is the only case where we can say anything non-trivial), ρ_0 might be possible to calculate.

It is clear that, if the given mixed manifold is trivial (i.e. if $V = B \times V_0$, with π being the projection to B), then the map ρ_0 is zero. The next talk aims to show that, in a certain sense, ρ indicates the non-triviality of V in a neighbourhood of V_0 .

Bibliography

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