Families of complex spaces and the foundations of analytic geometry

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Translator's note *This document is a translation into English of the following:*

> Douady, A. *Séminaire Henri Cartan* **13 (1)** (1960–61), Talks no. 2 ("Variétés et espaces mixtes"), 3 ("Déformations régulières"), and 4 ("Obstruction primaire à la déformation"). numdam.org/item/SHC_1960-1961__13_1

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[Translator] According to the complete list of talks, the notes from the first talk of the 1960/61 Séminaire Henri Cartan — "Fibrés en tores complexes" (also given by Adrien Douady) — were not copied, and thus seem to be lost to the past. What follows is a translation of the next three talks in this seminar series.

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2. Mixed manifolds and mixed spaces

I. Category of models

Let B be a topological space. We define the category $\mathscr{S}_B^n$ in the following manner: the objects of $\mathscr{S}_B^n$ are the open subsets of $B \times \mathbb{C}^n$, and a morphism $f : U \to U'$ from an open subset $U \subset B \times \mathbb{C}^n$ to an open subset $U' \subset B \times \mathbb{C}^n$ is a continuous map $f : U \to U'$ satisfying the following two conditions: *p. 2-01*

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ B & = & B \end{array}$$

commutes, where π_1 denotes the projection of $B \times \mathbb{C}^n$ to B; and

2. for all $x \in B$, the map $f_x : U_x \to U'_x$ is holomorphic, where

$$U_x = \{z \in \mathbb{C}^n \mid (x,z) \in U\}$$

(and similarly for U').

If B is endowed with the structure of a $\mathscr{C}^\infty$ manifold (resp. an $\mathbb{R}$-analytic manifold, resp. $\mathbb{C}$-analytic manifold), then we obtain a category $\mathscr{C}^\infty\mathscr{S}_B$ (resp. $\mathbb{R}\mathscr{S}_B$, resp. $\mathbb{C}\mathscr{S}_B$) by requiring the morphisms to be $\mathscr{C}^\infty$ (resp. $\mathbb{R}$-analytic, resp. $\mathbb{C}$-analytic).

More generally, if $f_1 : B \to B'$ is a continuous map from one topological space to another, then a *morphism of* $\mathscr{S}_{f_1}$ is a continuous map f from an object U of $\mathscr{S}_B$ to an object U' of $\mathscr{S}_{B'}$ such that

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ B & \xrightarrow[f_1]{} & B' \end{array}$$

commutes; and
2. $f_x : U_x \to U'_{f_1(x)}$ is holomorphic for all $x \in B$.

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If f_1 is a $\mathscr{C}^\infty$ map from one $\mathscr{C}^\infty$ manifold to another, then f will be a morphism of $\mathscr{C}^\infty\mathscr{S}_{f_1}$ if, further, it is a $\mathscr{C}^\infty$ map (resp. ...). We thus obtain, for every category of topological spaces, a fibred category $\mathscr{S}^n$ (resp. $\mathscr{C}^\infty\mathscr{S}^n$, resp. ...).

II. The definition of mixed spaces and mixed varieties

1. First definition

Let B and V be separated spaces, and let $\pi : V \to B$ be a continuous map. The structure of a *mixed space* over B is defined on V by a system of charts $\varphi_i : U_i \to V$, where the (U_i) are objects of $\mathscr{S}^n_B$; for each i, φ_i is a homeomorphism from U_i to an open subset of V such that the diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_i} & V \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & = & B \end{array}$$

commutes; finally, for all i and all j, the "change of chart" $\varphi_j^{-1} \circ \varphi_i$ is an isomorphism of $\mathscr{S}_B$ from an open subset of U_i to an open subset of U_j.

The structure thus defined is that of a $(\mathscr{C}^0,\mathbb{C})$*-mixed space*. If B is a $\mathbb{C}$-analytic space, and if the change of chart maps are all $\mathbb{C}$-analytic, then we have a $\mathbb{C}$*-analytic mixed space*. In this case, V itself is a $\mathbb{C}$-analytic space, and the fibres $V_x = \pi^{-1}(x)$ are $\mathbb{C}$-analytic submanifolds.

If B is a $\mathscr{C}^\infty$ manifold (resp. $\mathbb{R}$-analytic, resp. $\mathbb{C}$-analytic), and if the change of chart maps are all $\mathscr{C}^\infty$ (resp. ...), then we have a $(\mathscr{C}^\infty,\mathbb{C})$*-mixed manifold* (resp. $(\mathbb{R},\mathbb{C})$, resp. $(\mathbb{C},\mathbb{C})$). In this case, V itself is a manifold. Note that the notion of a $(\mathbb{C},\mathbb{C})$-mixed manifold, or a $\mathbb{C}$-analytic mixed manifold, reduces to simply having a $\mathbb{C}$-analytic manifold V endowed with a projection $\pi\colon V \to B$ onto another $\mathbb{C}$-analytic manifold such that π is of maximal rank at every point.[1]

Let $\pi\colon V \to B$ and $\pi'\colon V' \to B'$ be mixed spaces, and let $f_1\colon B \to B'$ be a continuous (resp. ...) map. Then a *morphism from V to V' over f_1* is a continuous map $f\colon V \to V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[\;f_1\;]{} & B' \end{array}$$

commutes, and such that, for any charts $\varphi_i\colon U_i \to V$ and $\varphi'_j\colon U'_j \to V'$, the map ${\varphi'_j}^{-1}\circ f\circ\varphi_i$ is a morphism of $\mathscr{S}_{f_1}$ (resp. ...) from an open subset of U_i to U_j. *p. 2-03*

2. An equivalent definition

We now give another way of defining mixed spaces, equivalent to the above.

Given separated spaces B and V, along with a continuous map $\pi\colon V \to B$, the structure of a *pre-mixed space* consists of the structure of a $\mathbb{C}$-analytic manifold on each fibre $V_x = \pi^{-1}(x)$. Given pre-mixed spaces $\pi\colon V \to B$ and $\pi'\colon V' \to B'$, along with a continuous map $f_1\colon B \to B'$, a *morphism of pre-mixed spaces over f_1* is a continuous map $f\colon V \to V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[\;f_1\;]{} & B' \end{array}$$

commutes and induces a $\mathbb{C}$-analytic map on each fibre.

A *mixed space* is a pre-mixed space $\pi\colon V \to B$ such that every point $y \in V$ admits a neighbourhood W in V that is isomorphic as a pre-mixed space to an open subset of $B \times \mathbb{C}^n$, via an isomorphism over the identity. The morphisms of mixed spaces are the same: mixed spaces form a *full subcategory*.

3. Deformations

A mixed space $\pi\colon V \to B$ is said to be *proper* if B is locally compact and the map π is proper (i.e. the inverse image of any compact subset is compact). If it is a mixed manifold, then we can show that it is a fibred manifold that is locally trivial with respect to the underlying $\mathscr{C}^\infty$ structure, but the previous talk shows that, in general, any two fibres are not isomorphic as $\mathbb{C}$-analytic manifolds.

Definition. Let V_0 be a compact $\mathbb{C}$-analytic manifold, B a locally compact space, and $b_0 \in B$. Then a *$\mathbb{C}$-analytic deformation of V_0 over* (B, b_0) consists of a proper $\mathbb{C}$-analytic mixed space $\pi\colon V \to B$ along with an isomorphism of $\mathbb{C}$-analytic manifolds $i\colon V_0 \to \pi^{-1}(b_0)$. *p. 2-04*

[1] *[Trans.] The more common modern nomenclature is to simply call such an object a family of complex manifolds.*

The goal of this seminar is the study, at least local, and an attempt at a classification of, $\mathbb{C}$-analytic deformations of a given compact $\mathbb{C}$-analytic manifold V_0.

Definition. Let V_0 be a compact $\mathbb{C}$-analytic manifold. A $\mathbb{C}$-*analytic deformation* $(\pi\colon V \to B, i\colon V_0 \to V)$ *of* V_0 is said to be *locally complete* if, for any other deformation $(\pi'\colon V' \to B', i'\colon V_0 \to V')$ of V_0, there exists a neighbourhood B'_1 of b'_0 in B', an analytic map $f_1\colon B'_1 \to B$ with $f_1(b'_0) \to b_0$, and a morphism of $\mathbb{C}$-analytic mixed spaces $f\colon \pi'^{-1}(B'_1) \to V$ over f_1 such that $f \circ i' = i$. The deformation is said to be *locally universal* is furthermore the germ of f_1 at b'_0 is determined uniquely by this condition.

It seems that every compact $\mathbb{C}$-analytic manifold V_0 admits a locally complete $\mathbb{C}$-analytic deformation, and a locally universal one if the group of automorphisms of V_0 is discrete.

III. Vector fields

1. Study on models

Let B be a space, U an object of $\mathscr{S}_B$ (i.e. an open subset of $B \times \mathbb{C}^n$), b_0 a point of B, and set $U_0 = \pi^{-1}(b_0)$.

A holomorphic field of tangent vectors on U_0 (i.e. a holomorphic map from U_0 to $\mathbb{C}^n$) is said to be a *vertical holomorphic field* on U_0. A *vertical holomorphic field on* U is a continuous (resp. …) map $\theta\colon U \to \mathbb{C}^n$ that induces a vertical holomorphic field on each fibre U_x. If $f\colon U \to U'$ is an isomorphism in $\mathscr{S}_B$, then the *transport* $f_*\theta$ *of* θ *by* f is defined by

$$f_*\theta(f(x,z)) = \mathrm{D}_2 f_{x,z} \cdot \theta(x,z)$$

where $\mathrm{D}_2 f_{x,z}$ is the linear map from $\mathbb{C}^n$ to itself that is tangent to f_x at the point $z \in U_x$. This is again a vertical holomorphic field, since it follows from a Cauchy integral that the matrix $\mathrm{D}f_{x,z}$ depends continuously on the pair (x,z).

Now suppose that B is a $\mathscr{C}^\infty$ manifold, just for simplicity, and let T_0 be the tangent space to B at b_0. A field of tangent vectors to U defined on U_0, i.e. a map $\omega\colon U_0 \to T_0 \times \mathbb{C}^n$, *p. 2-05*
is said to be a *projectable holomorphic field* if $\omega(b_0,z) = (t_0, \theta(z))$ (where $t_0 \in T_0$ is a vector that does not depend on z, called the *projection* of the field ω) and $\theta(z)$ is a holomorphic vector field. If B is a $\mathbb{C}$-analytic space, possibly with a singularity at b_0, then we give the same definition, but with T_0 then being the *Zariski* tangent space to B at b_0, i.e. the dual of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the ideal of germs at b_0 of holomorphic functions on B that vanish at b_0.

If $f\colon U \to U'$ is an isomorphism of $\mathscr{C}^\infty \mathscr{S}_B$ (resp. …), then then transport $f_*\omega$ is defined by

$$f_*\omega(f(b_0,z)) = \mathrm{D}f_{b_0,z}\omega(b_0,z)$$

where $\mathrm{D}f_{b_0,z}\colon T_0 \times \mathbb{C}^n \to T_0 \times \mathbb{C}^n$ is now the linear map that is tangent to f at the point (b_0,z). This is a projectable holomorphic field. Indeed, the matrix $\mathrm{D}f_{b_0,z}$ can be written as

$$\begin{pmatrix} I & 0 \\ \mathrm{D}_1 f & \mathrm{D}_2 f \end{pmatrix}$$

and

$$\mathrm{D}_1 f\colon T \to \mathbb{C}^n$$
$$\mathrm{D}_2 f\colon \mathbb{C}^n \to \mathbb{C}^n$$

both depend holomorphically on z (for $\mathrm{D}_1 f$, this follows from the fact that f_x is holomorphic for every x). By setting $f_*\omega(b_0, z') = (t_0, \theta'(z'))$, we have

$$\theta'(z') = \mathrm{D}_1 f_{b_0,z}(t_0) + \mathrm{D}_2 f_{b_0,z}(\omega(z))$$
$$\text{if } z' = f_{b_0}(z)$$

which shows that $f_*\omega$ is indeed a projectable holomorphic field.

A *projectable holomorphic field on U* is a $\mathscr{C}^\infty$ field of vectors tangent to U that induces a projectable holomorphic field on each fibre.

2. Vector fields on a mixed manifold

Let $\pi\colon V \to B$ be a $(\mathscr{C}^\infty, \mathbb{C})$-mixed manifold (resp. ..., resp. a $\mathbb{C}$-analytic mixed space). By transporting along the charts, we define the notions of

- vertical holomorphic fields on an open subset of a fibre;
- vertical holomorphic fields on a open subset of V;
- projectable holomorphic fields on an open subset of a fibre; and
- projectable holomorphic fields on an open subset of V.

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Let ξ be a $\mathscr{C}^\infty$ vector field (resp. ...) on V. By integrating ξ, we obtain a $\mathscr{C}^\infty$ map, denoted by e^ξ, from an open subset $W \subset \mathbb{R} \times V$ containing $\{0\} \times V$ (resp. $\mathbb{C}$-analytic map from an open subset $W \subset \mathbb{C} \times V$) to V, characterised by

1. $e^\xi(t_1 + t_2, y) = e^\xi(t_1, e^\xi(t_2, y))$, with the left-hand side being defined whenever the right-hand side is; and
2. $\frac{\partial}{\partial t} e^\xi(t, y)|_{0,y} = \xi(y)$.

Note that W is a mixed manifold over $\mathbb{R} \times B$ (resp. a mixed space over $\mathbb{C} \times B$).

Proposition. *For $e^\xi\colon W \to V$ to be a morphism of mixed spaces over the projection $\mathbb{R} \times B \to B$, it is necessary and sufficient for ξ to be a vertical holomorphic field. For $e^\xi\colon W \to V$ to be a morphism of mixed spaces over a map from an open subset of $\mathbb{R} \times B$ containing $\{0\} \times B$ to B, it is necessary and sufficient for ξ to be a projectable holomorphic field.*

The proof is left to the reader.

IV. The Spencer–Kodaira map

Let $\pi\colon V \to B$ be a mixed manifold (resp. a $\mathbb{C}$-analytic mixed space), $b \in B$, and $V_0 = \pi^{-1}(b_0)$. Let T_0 be the tangent space to B at b_0 (resp. the Zariski tangent space). We introduce the following sheaves on V_0:

- Θ_0: the sheaf of germs of vertical holomorphic fields on V_0 ;
- Π_0: the sheaf of germs of locally projectable holomorphic fields on V_0 ; and
- Λ_0: the sheaf $\pi^* T_0$, i.e. the sheaf of germs of locally constant maps from V_0 to T_0.

We have an exact sequence of sheaves on V_0

$$0 \to \Theta_0 \to \Pi_0 \to \Lambda_0 \to 0$$

that gives rise to the long exact sequence in cohomology

$$\ldots \to \mathrm{H}^0(V_0;\Pi_0) \to \mathrm{H}^0(V_0;\Lambda_0) \xrightarrow{\delta} \mathrm{H}^1(V_0;\Theta_0) \to \ldots.$$

We also have a canonical map p. 2-07

$$\iota\colon T_0 \to \mathrm{H}^0(V_0;\Lambda_0)$$

that is injective if V_0 is non-empty, and surjective if V_0 is connected.

Definition. The *Spencer–Kodaira map* is the composition

$$\rho_0 = \delta \circ \iota\colon T_0 \to \mathrm{H}^1(V_0;\Theta_0).$$

This map is an essential tool in the local study of deformations of $\mathbb{C}$-analytic varieties. Note that Θ_0 is exactly the sheaf of germs of holomorphic fields of tangent vectors to V_0, and thus depends only on V_0, while T_0 depends only on the base. Also, Θ_0 is a coherent analytic sheaf on V_0, and, if V_0 is compact, then $\mathrm{H}^1(V_0;\Theta_0)$ is a finite-dimensional vector space over $\mathbb{C}$ [1]. We thus see that, in this case (which is the only case where we can say anything non-trivial), ρ_0 might be possible to calculate.

It is clear that, if the given mixed manifold is trivial (i.e. if $V = B \times V_0$, with π being the projection to B), then the map ρ_0 is zero. The next talk aims to show that, in a certain sense, ρ indicates the non-triviality of V in a neighbourhood of V_0.

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3. Regular deformations

I. The map $\widetilde{\rho}$

All throughout this talk, B is a $\mathscr{C}^\infty$ manifold (resp. $\mathbb{R}$-analytic, resp. $\mathbb{C}$-analytic); $\pi\colon V \to B$ denotes a proper mixed manifold; b_0 is a point of B; and $V_0 = \pi^{-1}(b_0)$ is thus a compact $\mathbb{C}$-analytic manifold. p. 3-01

Let $\widetilde{\Theta}$ (resp. $\widetilde{\Pi}$) be the sheaf of germs of vertical holomorphic (resp. locally projectable holomorphic) vector fields on V. The quotient sheaf $\widetilde{\Lambda} = \widetilde{\Pi}/\widetilde{\Theta}$ is exactly the inverse image under π of the sheaf $\widetilde{T}$ of germs of $\mathscr{C}^\infty$ fields (resp. …) of tangent vectors on B.

For every open subset U of B, set $V_U = \pi^{-1}(U)$. The exact sequence

$$0 \to \widetilde{\Theta} \to \widetilde{\Pi} \to \widetilde{\Lambda} \to 0$$

of sheaves on V_U gives rise to a homomorphism

$$\widetilde{\rho}_U\colon \mathrm{H}^0(U;\widetilde{T}) \xrightarrow{\pi_*} \mathrm{H}^0(V_U;\widetilde{\Lambda}) \xrightarrow{\delta} \mathrm{H}^1(V_U;\widetilde{\Theta}).$$

Let $\mathrm{R}^1\pi_*\widetilde{\Theta}$ be the sheaf on B defined by the presheaf $U \mapsto \mathrm{H}^1(V_U;\widetilde{\Theta})$. Then $\widetilde{\rho}$ becomes a homomorphism of sheaves on B:

$$\widetilde{\rho}\colon \widetilde{T} \to \mathrm{R}^1\pi_*\widetilde{\Theta}.$$

In particular, we have a homomorphism

$$\widetilde{\rho}_0\colon \widetilde{T}_0 \to \mathrm{R}^1\pi_*\widetilde{\Theta} = \mathrm{H}^1(V_0;\widetilde{\Theta})$$

where $\widetilde{T}_0$ is the vector space of germs at b_0 of fields of tangent vectors to B. Finally, we have a commutative diagram *p. 3-02*

$$\begin{array}{ccc} \widetilde{T}_0 & \xrightarrow{\widetilde{\rho}_0} & \mathrm{H}^1(V_0;\widetilde{\Theta}) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ T_0 & \xrightarrow[\rho_0]{} & \mathrm{H}^1(V_0;\Theta_0) \end{array}$$

where ρ_0 is the Spencer–Kodaira map [**2**].

Theorem 1. *For the proper mixed manifold $\pi\colon V \to B$ to be locally trivial in a neighbourhood of the point $b_0 \in B$, it is necessary and sufficient for the map $\widetilde{\rho}_0\colon \widetilde{T}_0 \to \mathrm{H}^1(V_0;\widetilde{\Theta})$ to be zero.*

Proof. —

a. *(Necessity).* If $\pi\colon V \to B$ is locally trivial at b_0, then, for every open subset U of B over which V is trivial, we have $\widetilde{\Pi} = \widetilde{\Lambda} \oplus \widetilde{\Theta}$ on V_U, and so $\delta\colon \mathrm{H}^0(V_U;\widetilde{\Lambda}) \to \mathrm{H}^0(V_U;\widetilde{\Theta})$ is zero.

b. *(Sufficiency).* Let $(\eta_1,\ldots,\eta_p)$ be $\mathscr{C}^\infty$ vector fields (resp. …) on a neighbourhood of b_0 in B, such that $(\eta_1(b_0),\ldots,\eta_p(b_0))$ forms a basis of the tangent space T_0 to B at b_0. It then follows from the hypothesis that the map

$$\mathrm{H}^0(V_0;\widetilde{\Pi}) \to \mathrm{H}^0(V_0;\widetilde{\Lambda})$$

is surjective. So let $(\xi_1,\ldots,\xi_p)$ be projectable holomorphic vector fields on a neighbourhood of V_0 in V, that project to $(\eta_1,\ldots,\eta_p)$. Let f be the map defined on a neighbourhood of $\{0\} \times V_0$ in $\mathbb{R}^p \times V_0$ (resp. $\mathbb{C}^p \times V_0$) by

$$f(t_1,\ldots,t_p,y) = e^{\xi_1}(t_1, e^{\xi_2}(\ldots, e^{\xi_p}(t_p,y)\ldots)).$$

It follows from the proposition stated in [2, Section III.2] that f induces an isomorphism of mixed manifolds from $U \times V_0$ to $\pi^{-1}(f_1(U))$ over f_1, where U is a sufficiently small cubical neighbourhood of 0 in $\mathbb{R}^p$, and f_1 is the map from U to B defined by

$$f_1(t_1,\ldots,t_p) = e^{\eta_1}(t_1, e^{\eta_2}(\ldots, e^{\eta_p}(t_p,b_0)\ldots)),$$

which proves the theorem.

p. 3-03

□

II. The regular case

For all $b \in B$, set $V_b = \pi^{-1}(b)$. Consider the family $\{\mathrm{H}^1(V_b;\Theta_b)\}_{b\in B}$ of finite-dimensional $\mathbb{C}$-vector spaces, and, for all $b \in B$, the map

$$\varepsilon_b\colon \mathrm{H}^1(V_b;\widetilde{\Theta}) \to \mathrm{H}^1(V_b;\Theta_b).$$

For every open subset $U \subset B$, we have a map

$$\widetilde{\varepsilon}_U\colon \mathrm{H}^1(V_U;\widetilde{\Theta}) \to \prod_{b\in U} \mathrm{H}^1(V_b;\Theta_B)$$

that defines, by varying U, a homomorphism from the sheaf $\mathrm{R}^1\pi_*\widetilde{\Theta}$ to the sheaf Φ on B defined by $\Phi(U) = \prod_{b\in U} \mathrm{H}^1(V_b;\Theta_b)$.

Definition.

We say that the proper mixed manifold $\pi\colon V \to B$ is *regular* if

1. the dimension of $\mathrm{H}^1(V_b;\Theta_b)$ does not depend on the point $b \in B$; and
2. we can endow $E = \bigcup_{b\in B} \mathrm{H}^1(V_b;\Theta_b)$ with the structure of a $\mathscr{C}^\infty$ vector bundle (resp. ...) such that $\widetilde{\varepsilon}$ is an isomorphism from the sheaf $\mathrm{R}^1\pi_*\widetilde{\Theta}$ to the sheaf of germs of $\mathscr{C}^\infty$ sections (resp. ...) of the bundle E.

In fact, Kodaira and Spencer have shown [7] that, by identifying the H^1 spaces with spaces of harmonic forms, condition (2) is a consequence of condition (1).

Then Theorem 1 has the following corollary:

Proposition 1. *For the proper mixed manifold $\pi\colon V \to B$ to be locally trivial, it is necessary and sufficient for it to be regular and, for all $b \in B$, for the Spencer–Kodaira map*

$$\rho_b\colon T_b \to \mathrm{H}^1(V_b;\Theta_b)$$

to be zero.

Indeed, since $\widetilde{\varepsilon}$ is injective, this condition implies that the map

$$\widetilde{\rho}_b\colon \widetilde{T}_b \to \mathrm{H}^1(V_b;\widetilde{\Theta})$$

is zero for all b. *p. 3-04*

At the end of this talk, we will construct a counter-example which shows that it is necessary to assume that the mixed manifold is regular.

III. An example of non-regular deformation: Hopf manifolds

1. Hopf manifolds

Let $n \geqslant 2$ be an integer, and let b be an $(n\times n)$ matrix with coefficients in $\mathbb{C}$, whose eigenvalues are all of modulus >1. The free group $L(b)$ generated by b acts freely on $\widetilde{V} = \mathbb{C}^n \setminus \{0\}$, and the quotient space $\widetilde{V}/L(b)$, which we call the *Hopf manifold defined by* b, is a compact $\mathbb{C}$-analytic manifold that is homeomorphic to $S^{2n-1}\times S^1$.

Note that V_b and $V_{b'}$ are isomorphic if and only if there exists some a such that $b' = aba^{-1}$ or $b' = ab^{-1}a^{-1}$ (cf. Appendix).

Let Θ be the sheaf of germs of holomorphic fields of tangent vectors on V_b.

Proposition 2. *We can identify $\mathrm{H}^0(V_b;\Theta)$ with the vector space of matrices that commute with b, and $\mathrm{H}^1(V_b;\Theta)$ has the same dimension as this vector space.*

Proof. If X is a vector field on an open subset $U \subset \widetilde{V}$, then $b_*(X)$ is the vector field on the open subset $b(U)$ given by transporting via b, i.e. $b_*X(u) = bX(b^{-1}u)$. Let $\mathscr{U} = \{U_i\}$ be a cover of V by simply connected Stein open subsets; for all i, set $\widetilde{U}_i = \chi^{-1}\{U_i\}$, where χ is the canonical map from $\widetilde{V}$ to V_b. The cover $\widetilde{\mathscr{U}} = \{\widetilde{U}_i\}$ of $\widetilde{V}$ consists of Stein open subsets that are invariant under b (not necessarily connected, but this doesn't matter). Then b_* defines a map, again denoted by b_*, from the group of cochains $C^\bullet(\widetilde{V},\widetilde{U};\Theta)$ to itself.

Lemma 1. *We have the exact sequence*

$$0 \to C^\bullet(V_b,\mathscr{U};\Theta) \xrightarrow{\chi^*} C^\bullet(\widetilde{V},\widetilde{U};\Theta) \xrightarrow{1-b_*} C^\bullet(\widetilde{V},\widetilde{U};\Theta) \to 0.$$

Proof. The only thing that we need to verify is that the map $1-b_*$ is surjective. For all $(i_0,\ldots,i_q)$, let $U'_{i_0,\ldots,i_q}$ be an open subset of $\widetilde{V}$ such that *p. 3-05*

$$\chi\colon U'_{i_0,\ldots,i_q} \to U_{i_0,\ldots,i_q}$$

is a homeomorphism. The $\widetilde{U}_{i_0,\ldots,i_q}$ is a disjoint union of the $b_*^p U'_{i_0,\ldots,i_q}$, where $p\in\mathbb{Z}$, and every $\gamma\in C^q(\widetilde{V},\widetilde{U};\Theta)$ can be written in the form $\gamma=\gamma_1-\gamma_2$, with $\gamma_1=0$ on $b^p(U'_{i_0,\ldots,i_q})$ for $p<0$, and $\gamma_2=0$ for $p\geqslant 0$. Set

$$\beta = \sum_{p\geqslant 0} b_*^p\gamma_1 + \sum_{p<0} b_*^p\gamma_2$$

(which is a locally finite sum). Then $\beta - b_*\beta = \gamma$, whence Lemma 1. □

Now, to finish the proof of Proposition 2. From Lemma 1, we have the following exact sequence:

$$0\to \mathrm{H}^0(V_b;\Theta) \xrightarrow{\chi^*} \mathrm{H}^0(\widetilde{V};\Theta) \xrightarrow{1-b_*} \mathrm{H}^0(\widetilde{V};\Theta) \xrightarrow{\delta_*} \mathrm{H}^1(V_b;\Theta) \xrightarrow{\chi^*} \mathrm{H}^1(\widetilde{V} y\Theta) \xrightarrow{1-b_*} \mathrm{H}^1(\widetilde{V};\Theta).$$

We can show that

$$\chi^*\colon \mathrm{H}^1(V_b;\Theta)\to \mathrm{H}^1(\widetilde{V};\Theta)$$

is zero: if $n>2$, it is evident, since $\mathrm{H}^1(\widetilde{V};\Theta)=0$; if $n=2$, then a direct calculation on the cochains of a cover of $\widetilde{V}$ by two Stein open subsets shows that

$$1-b_*\colon \mathrm{H}^1(\widetilde{V};\Theta)\to \mathrm{H}^1(\widetilde{V};\Theta)$$

is bijective.

Now $\mathrm{H}^0(\widetilde{V};\Theta)$ is the space of holomorphic vector fields on $\widetilde{V}$, but such a field extends to a holomorphic vector field on $\mathbb{C}^n$, and $\mathrm{H}^0(\widetilde{V},\Theta)=L\oplus M$, where L is the space of fields of linear vectors, and M is the space of fields of second-order vectors at 0. The subspaces L and M are invariant under b_*, and $1-b_*\colon M\to M$ is an isomorphism. Then Proposition 2 follows from remarking that, if an element of L is represented by a matrix a, then $b_*a = bab^{-1}$. □

2. Mixed manifolds whose fibres are Hopf manifolds

Let B be the set of all $(n\times n)$ matrices with coefficients in $\mathbb{C}$ with eigenvalues all of modulus >1. This is an open subset of $\mathbb{C}^{n^2}$. Let α be the transformation from $B\times\widetilde{V}$ to itself defined by $\alpha(b,x)=(b,b(x))$. The free group $L(\alpha)$ generated by α acts linearly on $B\times\widetilde{V}$, and the quotient $V=B\times\widetilde{V}/L(\alpha)$ is a $\mathbb{C}$-analytic manifold. By endowing it with the projection $\pi\colon V\to B$ induced by the projection $\pi_1\colon B\times\widetilde{V}\to B$ after passing to the quotient, we obtain *p. 3-06* a $\mathbb{C}$-analytic mixed manifold that is proper, but not regular. Indeed, condition 1 of the definition of regular mixed manifolds is not satisfied: for example, for $n = 2$, the dimension of $\mathrm{H}^1(V_b;\Theta)$ is 4 if b is a scalar matrix, but 2 in all other cases.

Note that the dimension of $\mathrm{H}^1(V_b;\Theta_b)$ is an upper semi-continuous function of b, and that the set of b such that $\dim \mathrm{H}^1(V_b;\Theta_b) \geqslant k$ is a closed analytic subspace of B. This is a general result, that we hope to be able to prove in a later talk of this seminar.

3. Calculation of ρ

We have $T_b = \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) = L \subset \mathrm{H}^0(\widetilde{V};\Theta)$, and we defined, to prove Proposition 2, a surjective map $\delta_* : L \to \mathrm{H}^1(V_b;\Theta)$.

Proposition 3. *The Spencer–Kodaira map ρ is given, for the mixed manifold studied in this section, by*

$$\rho(a) = \delta_*(ab^{-1}).$$

In particular, it is surjective, and its kernel is the space of matrices of the form $[\ell,b]$ for $\ell \in L$.

Proof. Let $a \in T_b = L$. Let $\{U_i\}$ be a cover of V_b by simply connected Stein open subsets, and, for each i, let U_i' be a connected component of $\widetilde{U}_i$.

Let η_i' be the projectable holomorphic field on U_i' defined by $\eta_i'(x) = (a,0)$; let $\widetilde{\eta}_i$ be the projectable holomorphic field on $\widetilde{U}_i$ defined by $\widetilde{\eta}_i = \alpha_*^k \eta_i'$ on $b^k(U_i')$; and let η_i be the projectable holomorphic field on U_i corresponding to $\widetilde{\eta}_i$. By definition, $\rho(a)$ is the cohomology class of the cochain $\{\theta_{ij}\}$, where $\theta_{ij} = \eta_j - \eta_i$ is a vertical holomorphic field on U_{ij}. p. 3-07

Set $\widetilde{\eta}_i(x) = (a,\beta_i(x))$. Then $\beta \in C^0(\widetilde{V};\Theta)$, and we have $(1-b_*)\beta = ab^{-1} \in L \subset \mathrm{H}^0(\widetilde{V};\Theta)$. Indeed, $\alpha_*\eta = \eta$, $\alpha_*\eta_i(b_{-1}x) = \eta_i(x)$, and

$$\alpha_*(a,\beta(b^{-1}x)) = (a,\beta(x)),$$

whence

$$ab^{-1}x + b\cdot\beta(b^{-1}x) = \beta(x).$$

We thus deduce that $\theta = \delta_*(ab^{-1})$, which proves Proposition 3. ☐

4. A counter-example

Take $n = 2$, and $\sigma \in \mathbb{C}$ such that $|\sigma| > 1$. Let $B' \subset B$ be the set of matrices of the form

$$\begin{pmatrix} \sigma & t \\ 0 & \sigma \end{pmatrix}$$

where $t \in \mathbb{C}$, and let $V' = \pi^{-1}(B')$ be the mixed manifold induced by V over V'; now B' is a line, and its tangent space T_b' at b is generated, for all b, by $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It follows from Proposition 3 that the Spencer–Kodaira map

$$\rho' : T_b(B') \to \mathrm{H}^1(V_b;\Theta)$$

is zero if and only if

$$b \neq b_0 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

since, if $b \neq b_0$, then $a = [\ell, b]$, where $\ell = \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}$; and if $b = b_0$, then ρ' is injective.

We can also see that V' is trivial on $B' \setminus \{b_0\}$.

Let $\varphi\colon \mathbb{C} \to B' \subset B$ be the map defined by

$$\varphi(t) = \begin{pmatrix} \sigma & t^2 \\ 0 & \sigma \end{pmatrix}$$

and let V^φ be the mixed manifold given by the inverse image of V under φ. The Spencer–Kodaira map ρ_t^φ from $\mathbb{C}$ to $\mathrm{H}^1(V_{\varphi(t)};\Theta)$ is the composition *p. 3-08*

$$\rho'_{\varphi(t)} \circ \mathrm{D}\varphi\colon \mathbb{C} \to T'_{\varphi(t)} \to \mathrm{H}^1(V_{\varphi(t)};\Theta),$$

and this is zero for all t, since, if $t \neq 0$, then $\rho'_{\varphi(t)}$ is zero; and, if $t = 0$, then $\mathrm{D}\varphi$ is zero.

However, the mixed manifold V^φ is not locally trivial, since V_0^φ is not isomorphic to V_t^φ for $t \neq 0$.

5. Question (K. Srinivasacharyulu)

We know that the Hopf manifolds are non-K"{a}hler, and thus non-algebraic. For $n = 2$, the manifold V_b admits non-constant meromorphic functions if and only if b can be diagonalised with eigenvalues σ_1 and σ_2 satisfying $\sigma_1^p = \sigma_2^q$ for some integers p and q (and there is then the function $x_1^p x_2^{-q}$). The set of b satisfying this property is neither open nor closed, but it is a countable union of closed analytic subspaces. An analogous phenomenon arises for deformations of complex tori. Is this result general?

Appendix

With the notation of §III.1, let $f\colon V_b \to V_{b'}$ be an isomorphism of $\mathbb{C}$-analytic manifolds. This lifts to an isomorphism of universal coverings

$$\widetilde{f}\colon \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}.$$

By Hartog, $\widetilde{f}$ extends to an isomorphism $g\colon \mathbb{C}^n \to \mathbb{C}^n$. We necessarily have

$$g(bz) = (b')^k g(z) \tag{$*$}$$

where $z \in \mathbb{C}^n$, and k is an integer; the same property, applied to the inverse map of g, shows that $k = \pm 1$. Let a be the linear map that is tangent to g at the origin; the identity $(*)$ then gives

$$ab = (b')^k a$$
$$k = \pm 1$$

whence

$$b' = aba^{-1} \quad \text{or} \quad b' = ab^{-1}a^{-1}.$$

4. The primary obstruction to deformation

Introduction

p. 4-01

Let V_0 be a compact complex-analytic manifold, and let Θ be the sheaf of germs of holomorphic fields of tangent vectors. We ask the following question: given an element $a \in \mathrm{H}^1(V_0,\Theta)$, does there exists a deformation of V_0, with a non-singular base (i.e. a fibred mixed manifold $\pi\colon V \to B$, with $b_0 \in B$, along with an isomorphism $V_0 \xrightarrow{\cong} \pi^{-1}(b_0)$), such that a is the image, under the map ρ defined in [Talk no. 2], of a vector v that is tangent to B at b_0? An element $a \in \mathrm{H}^1(V_0,\Theta)$ for which the answer is positive is called a *deformation vector*. We will give a necessary condition for a to be a deformation vector; this condition is written $[a \smile a] = 0$. We will then give an example where this condition is not satisfied.

I. Exact sequences of sheaves of algebras

Let K be a commutative ring, and let Φ, Φ_1, and Φ_2 be sheaves of K-modules on some space X, and suppose that we have some given homomorphism $\Phi_1 \otimes \Phi_2 \to \Phi$, written as a product. We define, for any cover $\mathscr{U}$ of X, the *cup product*

$$\smile\colon C^p(X,\mathscr{U};\Phi_1)\otimes C^q(X,\mathscr{U};\Phi_2) \to C^{p+q}(X,\mathscr{U};\Phi)$$

by the formula

$$(\alpha\smile\beta)_{i_0,\ldots,i_{p+q}} = \alpha_{i_0,\ldots,i_p}\cdot\beta_{i_p,\ldots,i_{p+q}}.$$

We have the relation

$$\mathrm{d}(\alpha\smile\beta) = \mathrm{d}\alpha\smile\beta + (-1)^p\,\alpha\smile\mathrm{d}\beta.$$

This induces a cup product on the cohomology of the cover $\mathscr{U}$, and, by passing to the inductive limit over open covers, a cup product

$$\smile\colon \mathrm{H}^p(X;\Phi_1)\otimes\mathrm{H}^q(X;\Phi_2)\to\mathrm{H}^{p+q}(X;\Phi).$$

p. 4-02

Definition. A *sheaf of algebras* on X is a sheaf of modules Φ on X endowed with a product $\Phi\otimes\Phi\to\Phi$ (which we do not assume to be either commutative nor associative).

If $f\colon \Phi\to\Psi$ is a homomorphism of sheaves of algebras, then the kernel Φ' of f is a sheaf of two-sided ideals of Φ, i.e. we have products $\Phi'\otimes\Phi\to\Phi'$ and $\Phi\otimes\Phi'\to\Phi'$ such that the two diagrams

$$\begin{array}{ccc} \Phi'\otimes\Phi & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi\otimes\Phi & \longrightarrow & \Phi \end{array} \qquad \begin{array}{ccc} \Phi\otimes\Phi' & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi\otimes\Phi & \longrightarrow & \Phi \end{array}$$

both commute.

Proposition 1. *Let $0\to\Phi'\to\Phi\to\Phi''\to 0$ be an exact sequence of sheaves of algebras on X; let $a\in\mathrm{H}^p(X;\Phi'')$. Then $\delta a\in\mathrm{H}^{p+1}(X;\Phi')$, and, for any class $b\in\mathrm{H}^q(X;\Phi')$, we have $\delta a\smile b = 0$.*

Proof. Let $\mathscr{U}$ be a cover of X such that a and b are represented by cocycles α and β (respectively), and such that α lifts to a cochain $\eta \in C^p(X,\mathscr{U};\Phi)$. Then $\delta\eta$ is a cocycle in $C^{p+1}(X,\mathscr{U};\Phi')$ whose class in $\mathrm{H}^{p+1}(X;\Phi')$ is, by definition, δa, and $\delta a \smile b$ is the class of $\delta\eta \smile \beta$. But $\delta(\eta \smile \beta) = \delta\eta \smile \beta$, and $\eta \smile \beta$ is a cochain in $C^{p+q}(X,\mathscr{U};\Phi')$, since Φ' is a sheaf of ideals. So the cocycle $\delta\eta \smile \beta$ is cohomologous to 0 in $\mathrm{H}^{p+q+1}(X;\Phi')$, which proves the proposition. □

II. The primary obstruction

Let V_0 be a complex-analytic manifold, and Θ_0 the sheaf of germs of holomorphic fields of tangent vectors. Then Θ_0 is a sheaf of Lie algebras, and, if $a,b \in \mathrm{H}^\bullet(V_0,\Theta_0)$, then we denote by $[a \smile b]$ the cup product defined by the bracket $[-,-]\colon \Theta_0 \otimes \Theta_0 \to \Theta_0$. It satisfies

$$[b \smile a] = (-1)^{pq+1}[a \smile b]$$

for $a \in \mathrm{H}^p(V_0,\Theta_0)$ and $b \in \mathrm{H}^q(V_0,\Theta_0)$. *p. 4-03*

Theorem 1. *Let $\pi\colon V \to B$ be a mixed manifold, b_0 a point of B, $V_0 = \pi^{-1}(b_0)$, and let $\rho_0\colon T_0 \to \mathrm{H}^1(V_0,\Theta_0)$ be Spencer–Kodaira map. Then, if u and v are tangent vectors of B at b_0, we have*

$$[\rho_0(u) \smile \rho_0(v)] = 0.$$

Corollary. *Let V_0 be a complex-analytic manifold, and Θ the sheaf of germs of holomorphic fields of tangent vectors of V_0. If $a \in \mathrm{H}^1(V_0,\Theta)$ is a deformation vector, then*

$$[a \smile a] = 0.$$

Proof. (Proof of the Corollary). This is simply a particular case of Theorem 1; note that $[a \smile b]$ is a symmetric bilinear map from $\mathrm{H}^1 \otimes \mathrm{H}^1$ to H^2, and that we are in characteristic $0 \neq 2$. □

Proof. (Proof of Theorem 1). Consider the following sheaves on V_0:

- Θ_0: the sheaf of germs of vertical holomorphic fields on V_0;
- $\widetilde{\Theta}_0$: the sheaf of germs of vertical holomorphic fields on V;
- Π_0: the sheaf of germs of locally projectable holomorphic fields on V_0;
- $\widetilde{\Pi}_0$: the sheaf of germs of locally projectable holomorphic fields on V;
- Λ_0: the sheaf $\pi^* T_0$, where T_0 is the tangent space of B at b_0; and
- $\widetilde{\Lambda}_0$: the sheaf $\pi^*\widetilde{T}_0$, where $\widetilde{T}_0$ is the space of germs at b_0 of fields on B of tangent vectors of B.

We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\Theta}_0 & \longrightarrow & \widetilde{\Pi}_0 & \longrightarrow & \widetilde{\Lambda}_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \\
0 & \longrightarrow & \Theta_0 & \longrightarrow & \Pi_0 & \longrightarrow & \Lambda_0 & \longrightarrow & 0
\end{array}$$

whence we obtain the following commutative diagram: *p. 4-04*

$$\begin{array}{ccc} \widetilde{T}_0 & \xrightarrow{\widetilde{\rho}} & \mathrm{H}^1(V_0;\widetilde{\Theta}) \\ {\scriptstyle\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon} \\ T_0 & \xrightarrow[\rho]{} & \mathrm{H}^1(V_0;\Theta_0) \end{array}$$

Let $u,v \in T_0$ be fixed tangent vectors of B at b_0. We can always find vector fields $\widetilde{u}$ and $\widetilde{v}$ on B that take the values u and v (respectively) at b_0; $\epsilon(\widetilde{u}) = u$ and $\epsilon(\widetilde{v}) = v$. The exact sequence

$$0 \to \widetilde{\Theta}_0 \to \widetilde{\Pi}_0 \to \widetilde{\Lambda}_0 \to 0$$

is a sequence of homomorphisms of sheaves of Lie algebras, and so

$$[\widetilde{\rho}(\widetilde{u}) \smile \widetilde{\rho}(\widetilde{v})] = 0$$

by Proposition 1. But $\epsilon\colon \widetilde{\Theta}_0 \to \Theta_0$ is also a homomorphism of sheaves of Lie algebras, and the diagram

$$\begin{array}{ccc} \mathrm{H}^1(V_0,\widetilde{\Theta}_0) \otimes \mathrm{H}^1(V_0,\widetilde{\Theta}_0) & \xrightarrow{[-\smile-]} & \mathrm{H}^2(V_0,\widetilde{\Theta}_0) \\ {\scriptstyle\varepsilon\otimes\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon} \\ \mathrm{H}^1(V_0,\Theta_0) \otimes \mathrm{H}^1(V_0,\widetilde{\Theta}_0) & \xrightarrow[{[-\smile-]}]{} & \mathrm{H}^2(V_0,\Theta_0) \end{array}$$

commutes. We thus deduce that $[\rho(u) \smile \rho(v)] = 0$. □

p. 4-05

Remarks.

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1. We make essential use of the fact that $\epsilon\colon \widetilde{T}_0 \to T_0$ is surjective, and thus of the fact that B has no singularities.
2. We actually have $[\rho(u) \smile b] = 0$ for all $u \in T_0$, for any class $b \in \mathrm{H}^1(V_0,\Theta_0)$ that is in the image of $\mathrm{H}^1(V_0,\widetilde{\Theta}_0)$ under ϵ. In particular, for an element $a \in \mathrm{H}^1(V_0,\Theta_0)$ to be a regular deformation vector (in the sense of [Talk no. 3]), it is necessary and sufficient for $[a \smile b] = 0$ for all $b \in \mathrm{H}^1(V_0,\Theta_0)$.

If V_0 is a compact complex-analytic manifold, and $a \in \mathrm{H}^1(V_0,\Theta)$, then we call $[a \smile a] \in \mathrm{H}^2(V_0,\Theta)$ the *primary obstruction* to the deformation of V_0 along a. For a to be a deformation vector, it is necessary that this primary obstruction be zero; but it is not sufficient: we can define a sequence of set-theoretic maps ω_n, called *obstructions*, with $\omega_1\colon \mathrm{H}^1(V_0,\Theta) \to \mathrm{H}^2(V_0,\Theta)$ given by $\omega_1(a) = [a \smile a]$, and with ω_{k+1} defined on the subset of $\mathrm{H}^1(V_0,\Theta)$ where ω_k vanishes, with values in varying quotients[2] of $\mathrm{H}^2(V_0,\Theta)$, and a necessary condition for a to be a deformation vector is that all the $\omega_k(a)$ be defined and real. I do not know if *this* condition is sufficient. Kodaira, Spencer, and Nijenhuis [5] have shown that, if $\mathrm{H}^2(V_0,\Theta) = 0$, then every element of $\mathrm{H}^1(V_0,\Theta)$ is a deformation vector. In this case, we even have a locally universal deformation whose base is a manifold, and ρ is an isomorphism from the tangent space of this manifold to $\mathrm{H}^1(V_0,\Theta)$

[2] See the Appendix.

III. An example of obstruction

1. The manifold V_0

Let $X = E/\Gamma$ be a 2-dimensional complex torus, i.e. $E \cong \mathbb{C}^2$ and $\Gamma \cong \mathbb{Z}^4$, and let D the be projective line $\mathbb{P}^1\mathbb{C}$. Set $V_0 = X \times D$. The sheaf Θ of holomorphic fields of tangent vectors of V_0 is the direct sum of the sheaves of Lie algebras Θ_1 and Θ_2, where

$$\Theta_1 = \mathscr{O} \otimes_{\mathscr{O}_X} \pi_1^* \Theta_X$$
$$\Theta_2 = \mathscr{O} \otimes_{\mathscr{O}_D} \pi_2^* \Theta_D$$

where $\pi_1 \colon V_0 \to X$ and $\pi_2 \colon V_0 \to D$ are the projections, $\mathscr{O}$, $\mathscr{O}_X$, and $\mathscr{D}$ are the structure sheaves (sheaves of local rings), and Θ_X and Θ_D are the sheaves of germs of holomorphic fields of tangent vectors of X and D (respectively). We are mostly interested in Θ_2. Also, *p. 4-06* $\mathrm{H}^1(V_0, \Theta_2)$ is given by the Künneth exact sequence:

$$0 \to \mathrm{H}^0(X, \mathscr{O}_X) \otimes \mathrm{H}^1(D, \Theta_D) \to \mathrm{H}^1(V_0, \Theta_2) \to \mathrm{H}^1(X, \mathscr{O}_X) \otimes \mathrm{H}^0(D, \Theta_D) \to 0.$$

But we know that $\mathrm{H}^0(D, \Theta_D)$ is the Lie algebra $\mathfrak{a}$ of the group

$$A = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* = \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$$

of automorphisms of D, and that $\mathrm{H}^1(D, \Theta_D) = 0$, as we can easily see by taking a cover of D by two open subsets. We have already seen (in [Talk no. 1]) that, if $X = E/\Gamma$, then $\mathrm{H}^1(X, \mathscr{O}) = \mathrm{Hom}(\Gamma, \mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ is of dimension 2. So $\mathrm{H}^1(V_0, \Theta_2) = \mathrm{H}^1(X, \mathscr{O}) \otimes \mathfrak{a}$ is of dimension 6. The cup product

$$\mathrm{H}^1(V_0, \Theta_2) \otimes \mathrm{H}^1(V_0, \Theta_2) \to \mathrm{H}^2(V_0, \Theta_2)$$

is given by the formula

$$[(\gamma \otimes \alpha) \smile (\gamma' \otimes \alpha')] = (\gamma \smile \gamma') \otimes [\alpha, \alpha'].$$

The cone of elements $\varphi \in \mathrm{H}^1(V_0, \Theta_2)$ such that $[\varphi \smile \varphi] = 0$ can be identified with the cone of rank 1 tensors in $\mathrm{H}^1(X, \mathscr{O}) \otimes \mathfrak{a}$. Indeed, if $\varphi = \gamma \otimes \alpha$, then

$$[\varphi \smile \varphi] = (\gamma \smile \gamma) \otimes [\alpha, \alpha] = 0 \otimes 0 = 0$$

and, if φ is not a simple tensor, then we have

$$\varphi = \gamma \otimes \alpha + \gamma' \otimes \alpha'$$

with γ and γ' independent, and α and α' independent, so

$$[\varphi \smile] = 2(\gamma \smile \gamma') \otimes [\alpha, \alpha'] \neq 0.$$

2. The mixed space V

In this example, every element of $\mathrm{H}^1(V_0, \Theta_2)$ whose primary obstruction is zero is a deformation vector. More precisely: *p. 2-07*

Proposition 2.
There exists a mixed space $\pi \colon V \to B$ and a point $b_0 \in B$ such that

1. *$\pi^{-1}(b_0) = V_0$ (the manifold defined in §III.1);*
2. *there exists an isomorphism σ from a $\mathbb{C}$-analytic space B to the cone of elements $\varphi \in \mathrm{H}^1(V_0, \Theta_2)$ such that $[\varphi \smile \varphi] = 0$; and*
3. *for every subspace B' of B that has no singularities at b_0, the Spencer–Kodaira map ρ from the tangent space of B' at b_0 to $\mathrm{H}^1(V_0, \Theta)$ agrees with $\sigma\colon B' \to \mathrm{H}^1(V_0, \Theta_2)$.*

Let H be the analytic space of homomorphisms from Γ to $\mathfrak{a}$ whose images are contained in a vector subspace of $\mathfrak{a}$ that is 1-dimensional over $\mathbb{C}$ (i.e. (4×2) matrices of rank 1 with coefficients in $\mathbb{C}$). For every $h \in H$, $e \circ h$ is a homomorphism from Γ to A, where $e\colon \mathfrak{a} \to A$ denotes the exponential map, and we construct a manifold V_h that is fibred over X with fibre D as follows: V_h is the quotient of $E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (x, y) = (x + \gamma, ((e \circ h)(\gamma)) \cdot y).$$

These manifolds are the fibres of a mixed space $W \to H$, where W is the quotient of $H \times E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (h, x, y) = (h, x + y, (e \circ h(y)) \cdot y).$$

We now place the following equivalence relation on H: we have $h' \sim h$ if and only if $(h' - h)$ extends to an $\mathbb{C}$-linear map $f\colon E \to \mathfrak{a}$. Note that, if $h'(\Gamma)$ and $h(\Gamma)$ are contained in the same subspace L of $\mathfrak{a}$ of dimension 1 over $\mathbb{C}$ (or if $h' \sim h$), then we also have $f(E) \subset L$ (or $h \sim 0$ and $h' \sim 0$). In both cases, V_h and $V_{h'}$ are isomorphic, and we have an isomorphism $i_{h',h}\colon V_h \to V_{h'}$ defined by

$$i_{h',h}(x, y) = (x, e \circ f(x) \cdot y)$$

(in the first case), or

$$i_{h',h} = i_{h',0} \circ i_{0,h}$$

(in the second case). If h, h', and h'' are in the same class, then we have $i_{h''h} = i_{h''h'} \circ i_{h'h}$, p. 4-08
and we can place on W the equivalence relation

$$(h', z') \sim (h, z) \iff h' \sim h \text{ or } z' = i_{h'h} z$$

for $h, h' \in H$, $z \in V_h$, and $z' \in V_{h'}$.

Let B and V be the quotients of H and W (respectively) by these equivalence relations. We have a projection $V \to B$. To show that the structures of a $\mathbb{C}$-analytic space on H and W induce structures of a $\mathbb{C}$-analytic space on their quotients B and V, it suffices to remark that we can lift B to a analytic subspace of H: let, for example, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be a basis of Γ such that (γ_1, γ_2) is a basis of E over $\mathbb{C}$; then each class $b \in B$ contains exactly one element $h \in H$ such that

$$h(\gamma_1) = h(\gamma_2) = 0.$$

3. Calculating ρ_0

Let T be the Zariski tangent space of B at b_0, i.e. the dual of $\mathfrak{I}/\mathfrak{I}^2$, where $\mathfrak{I}$ is the ideal of germs at b_0 of analytic functions on B that are zero at b_0. Then T_0 can be identified with $\mathrm{Hom}(\Gamma, a)/\mathrm{Hom}_{\mathbb{C}}(E, a)$. Also,

$$\begin{aligned} \mathrm{H}^1(V_0, \Theta) &= \mathrm{H}^1(V_0; \Theta_1) \oplus \mathrm{H}^1(V_0; \Theta_2) \\ &= \left(\mathrm{H}^1(X; \mathscr{O}) \otimes E\right) \oplus \left(\mathrm{H}^1(X; \mathscr{O}) \otimes a\right), \end{aligned}$$

and the second term of this term can be identified with the quotient $\mathrm{Hom}(\Gamma,\alpha)/\mathrm{Hom}_{\mathbb{C}}(E,\alpha)$. We are going to show that the map $\rho_0\colon T_0 \to \mathrm{H}^1(V_0;\Theta)$ is exactly the canonical injection defined by these identifications.

Let $u \in T_0 = \mathrm{Hom}(\Gamma,\alpha)/\mathrm{Hom}(E,\alpha)$ be the class of an element $h \in \mathrm{Hom}(\Gamma,\alpha)$, which we suppose to be of rank 1. Then we can write h in the form $\eta\otimes\sigma$, where $\eta \in \mathrm{Hom}(\Gamma,\mathbb{C})$, $\sigma \in \alpha$, and we can consider h as a tangent vector to H at 0. Let $\overline{h}$ be the field of tangent vectors to $H\times E\times D$ at $0\times E\times D$ that projects onto h, and thus whose components over $E\times D$ are zero. Let (U_i) be a cover of $X = E/\Gamma$ by simply connected open subsets, and choose, for each *p. 4-09* i, a component $\widetilde{U}_i$ of the inverse image of U_i in E. We will denote by v_i the image over $U_i\times D$ of the field $\overline{h}|\widetilde{U}_i\times D$. This is a projectable holomorphic field on $0\times U_i\times D$ of tangent vectors of $H\times U_i\times D$, and we set $w_{ij} = v_j - v_i$, so that w_{ij} is a vertical holomorphic field on $U_{ij}\times D$, and these fields form a cocycle whose cohomology class will be, by definition, $\rho_0(u)$.

Let $x \in U_{ij}$, and let $\widetilde{x}_i$ and $\widetilde{x}_j$ be its inverse image in $\widetilde{U}_i$ and $\widetilde{U}_j$ (respectively). We have that $\widetilde{x}_j = \widetilde{x}_i + \gamma_{ij}(x)$, where $\gamma_{ij}(x)\in\Gamma$, and

$$w_{ij}(x) = \overline{h}(\widetilde{x}_j) - [\gamma_{ij}(x)]_*(\overline{h}(\widetilde{x}_i)) = -h(\gamma_{ij}(x)) \in \alpha.$$

Now w_{ij} is a vector field on D, and so

$$(w_{ij}) \in \mathrm{Z}^1(V_0,(U_i\times D);\Theta_2),$$

and w_{ij} is of the form $\zeta\otimes\alpha$, where $\zeta \in \mathrm{Z}^1(V_0,(U_i\times D);\mathscr{O})$ is the cocycle defined by $\zeta_{ij}(x) = -\eta(\gamma_{ij}(x))$. This is a cocycle whose cohomology class is (up to a sign) the element of $\mathrm{H}^1(V_0,\mathscr{O})$ that is identified with the class η in $\mathrm{Hom}(\Gamma,\mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E,\mathbb{C})$. QED.

Appendix: Higher obstructions

I. Definition of obstructions

1. The sheaf of germs of vertical automorphisms

Let V_0 be a $\mathbb{C}$-analytic manifold, which we assume to be compact, and B a $\mathbb{C}$-analytic space, and let $b_0 \in B$. We are going to define a sheaf Γ of non-abelian groups on V_0. For every *p. 4-10* open subset U of V_0, consider the isomorphisms of analytic varieties $\gamma\colon W \to W'$, where W and W' are open subsets of $B\times V_0$ that contain $\{b_0\}\times U$, such that the following conditions are satisfied:

1. $\pi_1\gamma = \pi_1$ is the projection $B\times V_0$ to B;
2. γ is the identity on $\{b_0\}\times U$.

Then $\Gamma(U)$ consists of equivalence classes of these isomorphisms, where we identify γ_1 with γ_2 if they agree on a neighbourhood of $\{b_0\}\times U$.

It is clear that $\Gamma(U)$ is a group under composition of isomorphisms, and that the $\Gamma(U)$ form a sheaf Γ of non-abelian groups.

Proposition 1. *We can identify* $\mathrm{H}^1(V_0,\Gamma)$ *with the set of classes of deformation germs of* V_0 *over* (B,b_0).

Recall that a deformation germ of V_0 over (B,b_0) is a deformation of V_0 over a neighbourhood of b_0 in B, and that two such deformations (B',b_0,V',π',ι') and $(B'',b_0,V'',\pi'',\iota'')$ are locally equivalent if there exists a neighbourhood W' of $(\pi')^{-1}(b_0)$ in V', a neighbourhood W'' of $(\pi'')^{-1}(b_0)$ in V'', and an isomorphism φ from W' to W'' such that the diagram

$$\begin{array}{ccc} V_0 & = & V_0 \\ \downarrow & & \downarrow \\ W' & \xrightarrow{\varphi} & W'' \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi''} \\ B & = & B \end{array}$$

commutes. *p. 4-11*

Proof. (Proof of Proposition 1). Let (B',b_0,V,π,ι) be a deformation of V_0$ over a neighbourhood V' of b_0 in B. Then we can find a cover $\{U_i\}$ of V_0 and a cover $\{W_i\}$ of a neighbourhood of $\iota(V_0)$ in V, along with isomorphisms $\{h_i\}$, where h_i is an isomorphism from a neighbourhood of $\{b_0\}\times U_i$ in $B\times V_0$ to W_i that agrees with ι on $\{b_0\}\times U_i$, and such that $\pi\circ h_i=\pi_1$.

Set $\gamma_{ij}=h_i^{-1}\circ h_j$. We can show that the γ_{ij} define an element of $\Gamma(U_i\cap U_j)$, and that $\gamma_{ij}\circ\gamma_{jk}=\gamma_{ik}$. The γ_{ij} thus form a cocycle $\gamma\in \mathrm{Z}^1(V_0,\{U_i\};\Gamma)$. Such a cocycle is said to be *associated to the deformation*. It will still be associated to the deformation if pass to a finer cover. Let (B',b_0,V',π',ι') be a deformation that is locally equivalent to the first, and let γ' be a cocycle associated to this deformation. We can suppose, by refining the covers if necessary, that the cocycles γ and γ' are defined with respect to the same cover $\{U_i\}$ of V_0. Let f be an isomorphism from a neighbourhood of $\iota(V_0)$ in V to a neighbourhood of $\iota'(V_0)$ in V'. Set $f_i=(h_i')^{-1}\circ f\circ h_i$. Then $f_i\in\Gamma(U_i)$, and

$$f_i\circ\gamma_{ij}=\gamma'_{ij}\circ f_j.$$

We thus conclude that the cocycles associated to a deformation form a cohomology class that depends only on the local class of the deformation.

Conversely, suppose we have a locally finite cover $\{U_i\}$ of V_0 and a cocycle $\gamma\in\mathrm{Z}^1(V_0,\{U_i\};\Gamma)$. Then γ_{ij} can be represented by an isomorphism from an open W_{ij} of $B\times V_0$ to another open W_{ji}, with the two open subsets both containing $\{b_0\}\times U_{ij}$. Pick a refinement $\{U'_i\}$ of the cover $\{U_i\}$, and take some neighbourhood B'' of b_0 in B small enough such that $B''\times U'_{ij}\subset W_{ij}$ for all (i,j), and such that the equality $\gamma_{ij}\circ\gamma_{jk}=\gamma_{ik}$ holds wherever it is defined in $B''\times U'_{ijk}$. We thus obtain a deformation V of V_0 on B'' by gluing the $B''\times U'_i$ via the γ_{ij}.

Finally, we can show that all the above does indeed define a bijection between the set of local classes of deformations of V_0 over (B,b_0) and $\mathrm{H}^1(V_0;\Gamma)$. □

2. Higher obstructions

p. 4-12

For every open subset $U\subset V_0$, the group $\Gamma(U)$ is naturally filtered: denote by $\mathscr{F}_k(U)$ the group of vertical automorphisms that are tangent to the identity up to order $k-1$. Then Γ becomes a filtered sheaf:

$$\Gamma=\mathscr{F}_1\supset\mathscr{F}_2\supset\ldots\qquad\text{and}\ \bigcap\mathscr{F}_k=\{0\}.$$

Set

$$\mathscr{Q}_k = \Gamma/\mathscr{F}_{k+1}$$
$$\mathscr{G}_k = \mathscr{F}_k/\mathscr{F})_{k+1} = \mathrm{Ker}(\mathscr{Q}_k \to \mathscr{Q}_{k-1}).$$

For all k, $\mathscr{G}_k$ is a sheaf of abelian groups, which we will write additively. If $B = \mathbb{C}$ and $b_0 = 0$ (we then speak of *the deformation in one parameter*), for all k, $\mathscr{G}_k$ can be identified with the sheaf Θ of germs of vector fields tangent to V_0. In the general case,

$$\mathscr{G}_k = \mathfrak{m}^k/\mathfrak{m}^{k+1} \otimes \Theta$$

where $\mathfrak{m}$ is the maximal ideal of the point b_0 in B.

Now, if $a \in \mathscr{F}_p$ and $b \in \mathscr{F}_q$, then the commutator $aba^{-1}b^{-1}$ is in $\mathscr{F}_{p+q}$, and this defines a map $\mathscr{G}_p \otimes \mathscr{G}_q \to \mathscr{G}_{p+q}$ which endows $\mathscr{G}_\bullet = \bigoplus \mathscr{G}_k$ with the structure of a sheaf of Lie algebras that is isomorphic to the tensor product of Θ with the graded algebra associated to the maximal ideal $\mathfrak{m}$ of b_0 in B filtered by powers.

The exact sequence of non-abelian groups

$$0 \to \mathscr{G}_{k+1} \to \mathscr{Q}_{k+1} \to \mathscr{Q}_k \to 0$$

in which $\mathscr{G}_{k+1}$ is a subgroup of $\mathscr{Q}_{k+1}$ contained in its centre gives rise [3] to an exact sequence of pointed sets p. 4-13

$$\mathrm{H}^1(V_0;\mathscr{Q}_{k+1}) \to \mathrm{H}^1(V_0;\mathscr{Q}_k) \xrightarrow{\delta_k} \mathrm{H}^2(V_0;\mathscr{G}_{k+1})$$

i.e. for an element $q \in \mathrm{H}^1(V_0,\mathscr{Q}_k)$ to be in the image of $\mathrm{H}^1(V_0;\mathscr{Q}_{k+1})$, it is necessary and sufficient for $\delta_k q = 0$ in $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$. A *necessary* condition for q to be in the image of $\mathrm{H}^1(V_0;\Gamma) \to \mathrm{H}^1(V_0;\mathscr{Q}_k)$ is thus $\delta_k q = 0$ in $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$.

Definition. Let $q \in \mathrm{H}^1(V_0;\mathscr{Q}_i)$, and let $k \geqslant i$. We define an *obstruction of order k of the element q* to be the direct image in $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$ under δ_k of the inverse image of q in $\mathrm{H}^1(V_0;\mathscr{Q}_k)$. It is thus a subset of $\mathrm{H}^2(V_0;\mathscr{G}_{k+1})$. The obstruction is said to be *trivial* if the identity element belongs to this subset. Being trivial is a necessary and sufficient condition for q to be in the image of $\mathrm{H}^1(V_0;\mathscr{Q}_{k+1})$, and a necessary condition for q to be in the image of $\mathrm{H}^1(V_0;\Gamma)$.

Warning. If q is not in the image of $\mathrm{H}^1(V_0,\mathscr{Q}_k)$, then its obstruction of order k is empty, and thus non-trivial.

This definition is used most of all in the case of deformations in one parameter ($B = \mathbb{C}$ and $b_0 = 0$), where $\mathscr{G}_{k+1} = \Theta$ for all k, and $\mathscr{Q}_1 = \mathscr{G}_1 = \Theta$. The successive obstructions of an element $a \in \mathrm{H}^1(V_0;\Theta)$ are thus subsets of $\mathrm{H}^2(V_0;\Theta)$, and for a to be a deformation vector, it must be the case that all of its obstructions are trivial. Indeed, the element of $\mathrm{H}^1(V_0;\Theta)$ that corresponds, under the identifications we have made ($\Theta = \mathscr{Q}_1 = \Gamma/\mathscr{F}_2$, and Proposition 1), to a deformation germ is exactly the image under the Spencer–Kodaira map ρ of the canonical basis vector of the tangent space to $\mathbb{C}$ at 0.

II. Calculation of obstructions

1. Relation to the sheaf Ω

From now on, we work in the case of deformations in one parameter, i.e. $B = \mathbb{C}$ and $b_0 = 0$.

Let Ω be the sheaf of universal enveloping algebras of the Lie algebras of the sheaf Θ (i.e. $\Omega(U)$ is the universal enveloping algebra of $\Theta(U)$). *p. 4-14*

Then Ω contains Θ as a subsheaf, and even as a direct factor (by the Poincaré–Birkhoff–Witt Theorem in characteristic 0). For all k, consider the sheaf of algebras $\Omega_k = \Omega[t]/(t^{k+1})$. For $i \leqslant k$, we have a map of sheaves of sets

$$\exp_i : \Theta \to \Omega_k$$

defined by

$$\exp_i(\Theta) = \sum_p \frac{1}{M} \Theta^p t^p$$

Proposition 2. (Campbell–Hausdorff). *We can identify $\mathscr{Q}_k$ with the sheaf of multiplicative subgroups of Ω_k generated by the images of the $\exp_i$ for $i \leqslant k$.*

The proof of this proposition will not be given here. We denote by $\Omega_k^\times$ the sheaf of multiplicative subgroups of Ω_k consisting of the elements whose constant terms is 1. The commutative diagram of sheaves of (non-abelian) groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta & \longrightarrow & \mathscr{Q}_{k+1} & \longrightarrow & \mathscr{Q}_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega & \longrightarrow & \Omega_{k+1}^\times & \longrightarrow & \Omega_k^\times & \longrightarrow & 0
\end{array}$$

gives rise to a commutative diagram of sets

$$\begin{array}{ccc}
\mathrm{H}^1(V_0;\mathscr{Q}_k) & \xrightarrow{\delta_k} & \mathrm{H}^2(V_0;\Theta) \\
\downarrow & & \downarrow \\
\mathrm{H}^1(V_0;\Omega_k^\times) & \xrightarrow[\delta_k]{} & \mathrm{H}^2(V_0;\Omega)
\end{array}$$

in which $\mathrm{H}^2(V_0;\Theta)$ is a vector subspace of $\mathrm{H}^2(V_0;\Omega)$.

2. Calculation of the primary obstruction

Now let $a \in \mathrm{H}^1(V_0;\Theta)$, and let $\alpha = (\alpha_{ij})$ be a cocycle of the class a (the choice of the cocycle α does not matter, since every cocycle that is cohomologous to a deformation cocycle is itself a deformation cocycle). The corresponding multiplicative cocycle in $\Omega_1^\times$ is $(1+\alpha_{ij}t)$. This cocycle can be lifted to $\Omega_i^\times$ as the cochain $(1+\alpha_{ij}t)$, and we have

$$\begin{aligned}
(1+\alpha_{ij}t)(1+\alpha_{jk}t) &= 1 + (\alpha_{ij}+\alpha_{jk})t + \alpha_{ij}\alpha_{jk}t^2 \\
&= (1+\alpha_{ik}t + \alpha_{ij}\alpha_{jk}t^2) \\
&= (1+\alpha_{ik}t)\{1+\alpha_{ij}\alpha_{jk}t^2).
\end{aligned}$$

Finally, let

$$\delta_1 a = a \smile a$$

where the cup product is taken in the sheaf of algebras Ω.

Note that, if we denote by $\bar{\smile}$ the cup product taken in the sheaf of algebras opposite to Ω, i.e. defined on the level of cochains by $(\alpha \bar{\smile} \beta)_{ijk} = \beta_{jk}\alpha_{ij}$, we always have that $a \bar{\smile} b = -b \smile a$ in cohomology.

Consequently,

$$[a \smile a] = (a \smile a) - (a \bar{\smile} a) = 2a \smile a$$

and $\delta_1 a = a \smile a = \frac{1}{2}[a \smile a]$. We thus recover, up to a factor of $\frac{1}{2}$, the obstruction defined earlier in this talk.

3. Calculation of the secondary obstruction

Now suppose that $a \smile a = 0$, so that we can find a cochain $\beta = (\beta_{ij})$ such that $\delta\beta + \alpha \smile \alpha = 0$, i.e.

$$\beta_{ik} = \beta_{ij} + \beta_{jk} + \alpha_{ij}\alpha_{jk}.$$

Then $(1 + \alpha_{ij}t + \beta ijt^2)$ is a cocycle in $\Omega_2^\times$, and we can choose the cochain β to be a cocycle in $\mathscr{Q}_2$. *p. 4-16*

This cocycle can be lifted to $\Omega_3^\times$ as the cochain $(1 + \alpha_{ij}t + \beta_{ij}t^2)$, and we have that

$$\begin{aligned} &(1 + \alpha_{ij}t + \beta_{ij}t^2)(1 + \alpha_{jk}t + \beta_{jk}t^2) \\ &= 1 + (\alpha_{ij} + \alpha_{jk})t + (\beta_{ij} + \beta_{jk} + \alpha_{ij}\alpha_{jk})t^2 + (\alpha_{ij}\beta_{jk} + \beta_{ij}\alpha_{jk})t^3 \\ &= (1 + \alpha_{ik}t + \beta ikt^2)(1 + (\alpha_{ij}\beta_{jk} + \beta_{ij}\alpha_{jk})t^3). \end{aligned}$$

The secondary obstruction of a is thus the cohomology class of the cocycle $(\alpha_{ij}\beta_{jk} + \beta_{ij}\alpha_{jk}) \in \mathrm{Z}^2(V_0;\Omega)$. This class depends on the choice of the cochain β: if we choose some other $\beta' = \beta + \theta$, where $\Theta \in \mathrm{Z}^1(V_0;\Theta)$, then the cocycle is modified by $\alpha \smile \theta + \theta \smile \alpha$, and its class by an element of $[a \smile \mathrm{H}^1(V_0;\Theta)]$. We recover the *Massey triple product* (a,a,a) taken in the algebra Ω, but with a slightly more restrictive indetermination.

We can try to calculate this secondary obstruction without leaving the sheaf Θ, but the calculations are then much more complicated: we must take a cochain $\beta = (\beta_{ij})$ such that $\delta\beta + \frac{1}{2}[a \smile a] = 0$. Then the secondary obstruction of α is the class of the cocycle

$$[\alpha_{ij}, \beta_{jk}] + \frac{1}{6}[[\alpha_{ij}, \alpha_{jk}], \alpha_i j + 2\alpha_{jk}].$$

The calculation done in the sheaf of enveloping algebras Ω can be generalised to obstructions of order r: we are led to determining, by induction, cochains ω_r such that

$$\begin{cases} \omega_1 = \alpha \\ \delta\omega_r + \sum_{p+q=r} \omega_p \smile \omega_q = 0 \\ 1 + \sum_{1 \leqslant p \leqslant r} \omega_p t^p \in \mathrm{C}^1(V_0;\mathscr{Q}_r) \end{cases}$$

4. Using spectral sequences

p. 4-17

Proposition 3. *Let $\varphi\colon V_0 \to X$ be an arbitrary map, which gives rise to a spectral sequence of graded Lie algebras*

$$\mathrm{H}^\bullet(X;\mathbb{R}^\bullet\varphi\Theta) \Rightarrow \mathrm{H}^\bullet(V_0;\Theta).$$

Let

$$a \in \mathrm{H}^1(X;\varphi_*\Theta) \subset \mathrm{H}^1(V_0;\Theta).$$

If the element

$$-\frac{1}{2}[a \smile a] \in \mathrm{H}^2(X;\varphi_*\Theta) = E_2^{2,0}$$

is non-zero, but is the image under the differential d_2 *of the spectral sequence of an element* $b \in E_2^{0,1}$*, then the image of the secondary obstruction of* a *in* $E_\infty^{1,1}$ *consists of the elements of the form* $[a,b]$*. In particular, if, for all* b *such that* $\mathrm{d}_2 b = -\frac{1}{2}[a,a]$*, we have that* $[a,b] \neq 0$*, then the secondary obstruction is non-trivial.*

Warning. However, if $[a,b]=0$ in $E^{1,1}$, then we can only say that the secondary obstruction comes from $E_\infty^{2,0}$, and if this group is non-zero, then we cannot conclude anything.

Proof. Let α be a cocycle on V_0 representing the class a. The element $b \in E_2^{0,1}$ can be represented by a cochain

$$\beta = (\beta_{ij}) \in \mathrm{C}^1(V_0;\Theta)$$

such that p. 4-16

$$\delta\beta + \frac{1}{2}[a \smile a] = 0.$$

We thus obtain a cochain

$$\beta' \in \mathrm{C}^1(V_0;\Omega)$$

such that

$$1 + \alpha t + \beta' t^2 \in \mathrm{C}^1(V_0;\mathscr{Q}_2)$$

by setting $\beta'_{ij} = \beta_{ij} + \frac{1}{2}\alpha_{ij}^2$; this cochain satisfies $\delta\beta' + \alpha \smile \alpha = 0$. But this new cochain represents, in the $E_2^{0,1}$ term of the spectral sequence of the sheaf Ω, the same element b as the cochain β, since it differs from it by a cochain that comes from X. The secondary obstruction is thus the class of the cocycle $\alpha \smile \beta' + \beta' \smile \alpha$, which represents in the $E^{1,1}$ term of the spectral sequence the element $[a,b]$. □

This proposition allows us to construct non-trivial examples of secondary obstructions. Consider the group N of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x,y,z \in \mathbb{C}$, and let $Y = N/\Gamma$, where Γ is the subgroup of N consisting of elements where $x,y,z \in \mathbb{Z} + i\mathbb{Z}$. Then Y is fibred over a complex torus of dimension two $T^2 \cong \mathbb{C}^2/\mathbb{Z}^4$. We find non-trivial secondary obstruction elements in $\mathrm{H}^1(V_0;\Theta)$, where V_0 is the product of Y with a projective line D. (We use the spectral sequence obtained by projecting onto $T^2 \times D$). This variety has a "versal" deformation whose Zariski tangent space of the base B can be identified via the Spencer–Kodaira map ρ with $\mathrm{H}^1(V_0;\Theta)$. Further, B has, at its base point b_0, a conic singularity of degree 3, whose equation is given by the secondary obstruction. p. 4-19

I do not know of any examples of non-trivial secondary obstructions on varieties V_0 that satisfy $\mathrm{H}^0(V_0;\Theta) = 0$, but some very likely exist.

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