Divisors in algebraic geometry

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Translator's note

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In the first part of this talk, we will prove a theorem of Serre on complete varieties [6], following the methods of Grothendieck [4]. The second part is dedicated to generalities on divisors. In the literature, we often call the divisors studied here "locally principal" divisors. p. 4-01

The algebraic spaces considered here are defined over an algebraically closed field K. By "variety," we mean an irreducible algebraic space. If X is an algebraic space, we denote by $\mathscr{O}(X)$, $\mathscr{R}(X)$, etc. (or simply $\mathscr{O}$, $\mathscr{R}$, etc.) the structure sheaf, of regular functions, etc. on X (to define $\mathscr{R}(X)$ we assume that X is a variety). By "coherent sheaf" on X, we mean a coherent sheaf of $\mathscr{O}$-modules on X.

1 Preliminaries

References: [4–6]

If M is a module over an integral ring A (commutative and with 1), then we say that an element $m \in M$ is a *torsion element* if there exists some non-zero $a \in A$ such that $a \cdot m = 0$. We say that M is a *torsion module* (resp. *torsion-free module*) if every element of M is a torsion element (resp. if $M \neq 0$ and no non-zero element of M is a torsion element). The torsion elements of M form a torsion submodule of M (denoted by $T(M)$); if $M \neq 0$, then $M/T(M)$ is a torsion-free module. If M is a torsion module of finite type over A, then the ideal $\operatorname{ann} M$ of A (the ideal of A given by the elements $a \in A$ such that $aM = 0$) is non-zero.

Let X be an algebraic space and $\mathscr{F}$ a sheaf of $\mathscr{O}$-modules on X. We define $\operatorname{supp}\mathscr{F}$ to be the set of points $x \in X$ such that $\mathscr{F}_x \neq 0$. If $\mathscr{F}$ is coherent, then $\operatorname{supp}\mathscr{F}$ is a closed subset of X. If X is affine, then $\operatorname{supp}\mathscr{F}$ is the set defined by the ideal $\operatorname{ann}\mathrm{H}^0(X,\mathscr{F})$ of the affine algebra $\mathrm{H}^0(X,\mathscr{O})$, where $\mathrm{H}^0(X,\mathscr{F})$ is considered as a module over $\mathrm{H}^0(X,\mathscr{O})$.

A sheaf $\mathscr{F}$ of $\mathscr{O}$-modules on a *variety* X is said to be a *torsion sheaf* (resp. *torsion-free sheaf*) if, for every $x \in X$, the module $\mathscr{F}_x$ over the ring $\mathscr{O}_x$ is a torsion module (resp. torsion-free module). *p. 4-02*

Proposition 1. *If $\mathscr{F}$ is a coherent sheaf on a variety X, then there exists a coherent subsheaf $T(\mathscr{F})$ of $\mathscr{F}$ (and only one) such that $(T(\mathscr{F}))_x = T(\mathscr{F}_x)$.*

Proof. The uniqueness is trivial. The exists is a consequence of the fact that, if X is affine, then $T(\mathscr{F}_x)$ is given by localisation of the module $T(\mathrm{H}^0 *(X,\mathscr{F}))$ with respect to the maximal ideal of $\mathrm{H}^0(X,\mathscr{O})$ that defines x. □

C.*orollary If $\mathscr{F} \neq 0$ then $\mathscr{F}/T(\mathscr{F})$ is a torsion-free coherent sheaf.*[1]

Proposition 2. *If $\mathscr{F}$ is a coherent sheaf on the variety X, then* $\operatorname{supp}\mathscr{F} \neq X$ *if and only if $\mathscr{F}$ is a torsion sheaf.*

Proof. This is a trivial consequence of the fact that, if U is an affine open subset, then $\operatorname{supp}\mathscr{F} \cap U$ is defined by the ideal $\operatorname{ann}\mathrm{H}^0(U,\mathscr{F})$ of $\mathrm{H}^0(U,\mathscr{O})$, where $\mathrm{H}^0(U,\mathscr{F})$ is considered as a module over $\mathrm{H}^0(U,\mathscr{O})$. □

Proposition 3. *If $\mathscr{F}$ is a torsion-free coherent sheaf on a variety X, with $\mathscr{F} \subset \mathscr{R}^n$, then there exists a coherent sheaf $\mathscr{I} \neq 0$ of ideals of $\mathscr{O}$ such that $\mathscr{I} \cdot \mathscr{F} \subset \mathscr{O}^n$.*

Proof. Let $\mathscr{I}_x$ be the ideal $[\mathscr{O}_X^n : \mathscr{F}_x]$ of $\mathscr{O}_x$, i.e. the ideal of elements i_x of $\mathscr{O}_x$ such that $i_x\mathscr{F}_x \subset \mathscr{O}_x^n$. Since $\mathscr{F}_x$ is of finite type over $\mathscr{O}_x$, we know that $\mathscr{I}_x \neq 0$. If we take an affine open subset U of X, then we can prove that $\mathscr{I}_x$ is given by localisation of the ideal $[\mathrm{H}^0(U,\mathscr{O}^n) : \mathrm{H}^0(U,\mathscr{F})]$ of $\mathrm{H}^0(U,\mathscr{O})$ by the maximal ideal of $\mathrm{H}^0(U,\mathscr{O})$ that defines x. Thus $\{\mathscr{I}_x\}_{x\in X}$ defines a coherent sheaf $\mathscr{I}$ of ideals of $\mathscr{O}$ such that $\mathscr{I} \cdot \mathscr{F} \subset \mathscr{O}^n$. □

Let $\mathscr{F}$ be a torsion-free coherent sheaf on a variety X. Then the canonical homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}} \mathscr{R}$ is injective. The sheaves $\mathscr{R}$ and $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{R}$ are locally constant sheaves, and thus constant ([5]). We can then identify $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{R}$ with a vector space of finite dimension over $\mathscr{R}$ (we identify the field of rational functions with the sheaf $\mathscr{R}$ since $\mathscr{R}$ is constant). We call this dimension the *rank of* $\mathscr{F}$, and we can then consider $\mathscr{F}$ as a subsheaf of $\mathscr{R}^n$, where $n = \operatorname{rank}\mathscr{F}$.

Proposition 4. *Under the same hypotheses as in Proposition 3, there exists a coherent sheaf $\mathscr{I} \neq 0$ of ideals of $\mathscr{O}$ such that $\mathscr{I} \cdot \mathscr{F} \subset \mathscr{O}^n$, where $n = \operatorname{rank}\mathscr{F}$; then $\mathscr{O}^n/(\mathscr{I} \cdot \mathscr{F})$ and $\mathscr{F}/(\mathscr{I} \cdot \mathscr{F})$ are torsion sheaves.*

Proof. The proof is immediate. □

If Y is a closed subset of an algebraic space X, then we denote by $\mathscr{I}_Y$ the coherent sheaf of ideals of $\mathscr{O}$ defined by Y. *p. 4-03*

Proposition 5. *Let Y be a closed subset of an algebraic space X, and $\mathscr{F}$ a coherent sheaf on X, with $\operatorname{supp}\mathscr{F} \subset Y$; then there exists an integer k such that $\mathscr{I}_Y^k \mathscr{F} = 0$.*

Proof. We can reduce to the case where X is affine, since there exists a finite cover of X by affine opens. In this case, the hypothesis implies that the set defined by the ideal $\operatorname{ann}\mathrm{H}^0(X,\mathscr{F})$ is contained in Y. This implies, as is well known, that $\operatorname{ann}\mathrm{H}^0(X,\mathscr{F}) \supset \mathscr{I}_Y^k$. □

Proposition 6. *Let $\mathscr{F}$ be a coherent sheaf of fractional ideals on a variety X (i.e. a coherent subsheaf of $\mathscr{R}$) such that, for every x outside of a closed subset Y of X, $\mathscr{F}_x$ is an ideal of $\mathscr{O}_x$. Then there exists an integer k such that $\mathscr{I}_Y^k \cdot \mathscr{F} \subset \mathscr{O}$.*

Proof. By Proposition 3 and the hypothesis, there exists a coherent sheaf $\mathscr{J}$ of ideals of $\mathscr{O}$ such that $\mathscr{J}_x = \mathscr{O}_x$ if $x \notin Y$, and such that $\mathscr{J} \cdot \mathscr{F} \subset \mathscr{O}$. Thus $\operatorname{supp}(\mathscr{O}/\mathscr{J}) \subset Y$, and, by Proposition 5, there exists an integer k such that $\mathscr{I}_Y^k(\mathscr{O}/\mathscr{J}) = 0$. This implies that $\mathscr{I}_Y^k \subset \mathscr{J}$. □

2 Dévissage theorem

Let $\mathscr{C}$ be an abelian category, and $\mathscr{C}'$ a subcategory of objects of $\mathscr{C}$. We say that $\mathscr{C}'$ is *left exact in* $\mathscr{C}$ if[2]

1. every subobject of an object of $\mathscr{C}'$ is in $\mathscr{C}'$;
2. for every exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ in $\mathscr{C}$, the object $\mathscr{A}$ is in $\mathscr{C}'$ if the other two objects are in $\mathscr{C}'$.

Let X be an algebraic space. We denote by $\mathscr{C}(X)$ the abelian category of coherent sheaves on X. If Y is a closed subset of X, then a coherent sheaf on Y has a canonical extension to a coherent sheaf on X (extending by 0 outside of Y), and so we can consider $\mathscr{C}(Y)$ as a subcategory of $\mathscr{C}(X)$. With this notation, we have the following theorem: *p. 4-04*

Theorem (Dévissage). *Let $\mathscr{D}$ be a left-exact subcategory of $\mathscr{C}(X)$ that has the following property: for every closed irreducible subset Y of X, there exists a coherent sheaf $\mathscr{M}_Y$ of $\mathscr{C}(Y)$ that belongs to $\mathscr{D}$, and that is torsion-free as a sheaf on Y. Then $\mathscr{D} = \mathscr{C}(X)$.*

[1] *[Trans.] The condition that $\mathscr{F} \neq 0$ is unnecessary, but we include it here since it is in the original. Note that the zero sheaf is indeed a torsion-free sheaf, otherwise any coherent torsion sheaf $\mathscr{F}$ provides a counterexample to this corollary.*

[2] The axioms here that define a left-exact subcategory are slightly stronger than those of Grothendieck [4].

Proof. The proof works by induction on the dimension of X. If $\dim X = 0$, then X consists of a finite number of points $P_1, \ldots, P_r$, and a coherent sheaf on X can be identified with a system $\{N_i\}_{i=1,\ldots,r}$, where N_i is a vector space of finite dimension over K. Thus the sheaf $\mathscr{M}_{P_i}$ on P_i that we have, by hypothesis, is a vector space of finite dimension over K. By the axioms of a left-exact subcategory, it is trivial to show that every system $\{N_i\}_{i=1,\ldots,r}$, where N_i is a vector space of finite dimension over K, considered as a coherent sheaf on X, belongs to $\mathscr{D}$.

Now assume that we have proven the theorem for all dimensions $\leqslant (n-1)$. Let $\dim X = n$. Let Y be a closed subset of X such that $\dim Y \leqslant (n-1)$. We can easily show that $\mathscr{D} \cap \mathscr{C}(Y)$ is a left-exact subcategory of $\mathscr{C}(Y)$ that satisfies the hypotheses of the theorem. So, by the induction hypothesis, $\mathscr{D} \supset \mathscr{C}(Y)$.

We will now prove that, if $\mathscr{F}$ is a coherent sheaf on X with $\operatorname{supp} \mathscr{F} = Y$, then $\mathscr{F} \in \mathscr{D}$. If $\mathscr{I}_Y \cdot \mathscr{F} = 0$, then $\mathscr{F} \in \mathscr{C}(Y)$, and, by the above, $\mathscr{F} \in \mathscr{D}$. No matter what, by Proposition 5, there exists an integer $k \geqslant 1$ such that $\mathscr{I}_Y^k \mathscr{F} = 0$. We will complete the proof by induction on k. Suppose that that claim has been proven for every coherent sheaf $\mathscr{G}$ on X such that $\mathscr{I}_Y^{k-1} \mathscr{G} = 0$. For $\mathscr{F}$, we have an exact sequence

$$0 \to \mathscr{I}_Y \cdot \mathscr{F} \to \mathscr{F} \to \mathscr{F}/(\mathscr{I}_Y \cdot \mathscr{F}) \to 0.$$

The sheaf $\mathscr{I}_Y \mathscr{F}$ is annihilated by $\mathscr{I}_Y^{k-1}$, and the sheaf $\mathscr{F}/(\mathscr{I}_Y \mathscr{F})$ is annihilated by $\mathscr{I}_Y$. Thus $\mathscr{I}_Y \mathscr{F}$ and $\mathscr{F}/(\mathscr{I}_Y \mathscr{F})$ belong to $\mathscr{D}$. This implies that $\mathscr{F} \in \mathscr{D}$.

Suppose that X is a variety, and that $\mathscr{F}$ is a torsion-free sheaf on X. We can consider $\mathscr{F}$ as a coherent subsheaf of $\mathscr{R}^n$, where $n = \operatorname{rank} \mathscr{F}$, and, by Proposition 4, there then exists a coherent sheaf of ideals $\mathscr{I}$ such that $\mathscr{I} \cdot \mathscr{F} \subset \mathscr{O}^n$, and such that the sheaves $\mathscr{F}/(\mathscr{I}\mathscr{F})$ and $\mathscr{O}^n/(\mathscr{I}\mathscr{F})$ are torsion sheaves. Since $\mathscr{F}/(\mathscr{I}\mathscr{F})$ is a torsion sheaf, $\mathscr{F}/(\mathscr{I}\mathscr{F}) \in \mathscr{D}$; thus $\mathscr{F} \in \mathscr{D}$ if and only if $\mathscr{I}\mathscr{F} \in \mathscr{D}$. Analogously, $\mathscr{I}\mathscr{F} \in \mathscr{D}$ if and only if $\mathscr{O}^n \in \mathscr{D}$, and, by the axioms of an exact subcategory, if and only if $\mathscr{O} \in \mathscr{D}$. Thus $\mathscr{F} \in \mathscr{D}$ if and only if $\mathscr{O} \in \mathscr{D}$. If we repeat the same argument for the torsion-free sheaf $\mathscr{M}_X$, which we have by hypothesis, then we see that $\mathscr{O} \in \mathscr{D}$, which implies that $\mathscr{F} \in \mathscr{D}$. *p. 4-05*

Suppose again that X is a variety, but now that $\mathscr{F}$ is an arbitrary coherent sheaf. We will show that $\mathscr{F} \in \mathscr{D}$. We can assume that $\mathscr{F} \neq 0$, and we then have

$$0 \to T(\mathscr{F}) \to \mathscr{F} \to \mathscr{F}/T(\mathscr{F}) \to 0$$

where $T(\mathscr{F})$ is a torsion sheaf, and $\mathscr{F}/T(\mathscr{F})$ is a torsion-free sheaf. By Proposition 2, $\operatorname{supp} T(\mathscr{F}) \neq X$, and, since X is a variety, $\dim \operatorname{supp} T(\mathscr{F}) < \dim T(X)$. We then have, by the induction hypothesis, that $T(\mathscr{F}) \in \mathscr{D}$, and we have just proven that $\mathscr{F}/T(\mathscr{F}) \in \mathscr{D}$. Thus $\mathscr{F} \in \mathscr{D}$.

Now let X be an arbitrary algebraic space, and $X_1, \ldots, X_p$ its irreducible components. If $\mathscr{F}$ is a coherent sheaf on X, then $\mathscr{F}/(\mathscr{I}_{X_i}\mathscr{F})$ can be identified with a sheaf on the variety X_i (where $\mathscr{I}_{X_i}$ is the sheaf of ideals of $\mathscr{O}(X)$ determined by X_i), and, by the above, $\mathscr{F}/(\mathscr{I}_{X_i}\mathscr{F}) \in \mathscr{D}$. Thus the sheaf $\mathscr{G} = \sum_{i=1}^p \mathscr{F}/(\mathscr{I}_{X_i}\mathscr{F})$ belongs to $\mathscr{D}$. We have a canonical homomorphism $\varphi\colon \mathscr{F} \to \mathscr{G}$. The image of φ is a coherent subsheaf of $\mathscr{G}$, and so the image of φ belongs to $\mathscr{D}$.

We know that $\operatorname{supp} \operatorname{Ker} \varphi \subset \bigcup_{i \neq j} X_i \cap X_j$, and so $\dim \operatorname{supp} \operatorname{Ker} \varphi < \dim X$, and, by the induction hypothesis, $\operatorname{Ker} \varphi \in \mathscr{D}$. Thus $\mathscr{F} \in \mathscr{D}$, and the theorem is proven. □

Corollary (Serre's Theorem). *If $\mathscr{F}$ is a coherent sheaf on a complete algebraic space X, then $\mathrm{H}^0(X,\mathscr{F})$ is a vector space of finite dimension over K.*

Proof. We take $\mathscr{D}$ to be the category of all coherent sheaves $\mathscr{F}$ on X such that $\mathrm{H}^0(X,\mathscr{F})$ is of finite dimension over K. We can prove that $\mathscr{D}$ is a left-exact subcategory of $\mathscr{C}(X)$. Also, we know that, if Y is an irreducible closed subset of X, then Y is a complete variety. Thus the coherent sheaf $\mathscr{O}(Y)$ on Y is a torsion-free sheaf with the property that $\mathrm{H}^0(Y,\mathscr{O}(Y)) \cong K$, and so $\mathrm{H}^0(X,\mathscr{O}(Y)) = \mathrm{H}^0(Y,\mathscr{O}(Y))$ is of finite dimension over K (we denote also by $\mathscr{O}(Y)$ the canonical extension of $\mathscr{O}(Y)$ to X). By the Theorem, the corollary is proven. □

3 Divisors (Generalities)

Let X be an algebraic variety, and $\mathscr{R}^\times(X)$ and $\mathscr{O}^\times(X)$ (or simply $\mathscr{R}^\times$ and $\mathscr{O}^\times$) the constant sheaf on X of non-zero rational functions and the sheaf on X of invertible regular functions (respectively). The sheaves $\mathscr{R}^\times$ and $\mathscr{O}^\times$, endowed with their multiplicative structure, are sheaves of abelian groups. *p. 4-06*

A *divisor* D on X is a section of the quotient sheaf $\mathscr{R}^\times/\mathscr{O}^\times$. An element of $\mathscr{R}^\times$ that is a representative of the value $D(x)$ of D at x is called a *definition function of D at x*. More generally, a function $f \in \mathscr{R}^\times$ is called a *definition function of D in an open subset U* if, for all $x \in U$, f is a representative of $D(x)$; then f is determined up to an invertible regular function on U. Since we can locally lift a section of $\mathscr{R}^\times/\mathscr{O}^\times$ to a section of $\mathscr{R}^\times$, a divisor D is determined by the following data: a cover $\{U_i\}$ by open subsets, and non-zero rational functions f_i on U_i such that, on $U_i \cap U_j$, $f_{ij} = f_i/f_j$ is an invertible regular function. We have that $f_{ij}f_{jk}f_{ki} = 1$ on $U_i \cap U_j \cap U_k$, and, as is well known, this allows us to construct a locally trivial fibre bundle with $K^\times$ as the structure group; it is easy to see that this fibre bundle is determined up to equivalence [7]. We also know that the coherent sheaves of fractional ideals (i.e. the coherent subsheaves of $\mathscr{R}$) that are generated by f_i and f_j (respectively) agree on $U_i \cap U_j$, and do not depend on the choice of definition functions of D in U_i and U_j. This implies that the divisor D canonically determines a coherent sheaf of locally principal fractional ideals. We can easily see that the converse is true, and this gives us an equivalent definition of a divisor [1].

A divisor D is said to be *positive* if, for each $x \in X$, $D(x) \in \mathscr{O}_x/\mathscr{O}_x^\times$ (i.e. if all the definition functions of D at x are regular functions in x).

Since $\mathscr{R}^\times/\mathscr{O}^\times$ is a sheaf of abelian groups on X, there is a canonical structure of an abelian group on the set of divisors on X; this group is called the *group of divisors on X*. The composition law in this group is written additively, and the identity element in this group is thus called the *zero divisor*, and is denoted by (0).

If f is a non-zero rational function on X, then it defines a divisor $\operatorname{div} f$ by the data $(\operatorname{div} f)(x) = \operatorname{Im} f \subseteq \mathscr{R}^\times/\mathscr{O}_x^\times$. The divisors obtained in this way are called *principal divisors*, and form a subgroup of the group of divisors on X; the quotient group is called the *group of classes of divisors on X*. Two divisors D_1 and D_2 are said to be equivalent if they are equivalent module the group of principal divisors; we write $D_1 \sim D_2$. We have seen that a divisor defines, up to equivalence, a locally trivial algebraic fibre bundle with structure group $K^\times$. On the other hand, it is easy to see that a locally trivial algebraic fibre bundle with $K^\times$ as its structure group defines, up to equivalence, a divisor [7]. Thus the group of *p. 4-07* classes of divisors on X is equal to $\mathrm{H}^1(X, \mathscr{O}^\times)$, the group of classes of equivalent algebraic fibre bundles with $K^\times$ as their structure group.

We can define, in an analogous way, an *additive divisor* on a variety X as a section of the sheaf $\mathscr{R}/\mathscr{O}$ (the divisors defined above are called *multiplicative divisors*, or simply *divisors*). The additive divisors form an abelian group, and even a vector space over K. An additive divisor is determined by the following data: a cover $\{U_i\}$ of X by open subsets, and rational functions f_i on U_i such that $f_{ij} = f_i - f_j$ is a regular function on $U_i \cap U_j$. We can define, as for (multiplicative) divisors, the notions of definition functions of an additive divisor, equivalence between two additive divisors, etc. We find, for example, that $\mathrm{H}^1(X, \mathscr{O})$ is equal to the group of classes of additive divisors on X.

Let D be a (multiplicative) divisor on X. We define $\operatorname{supp} D$ to be the set of points $x \in X$ such that $D(x)$ is not the identity element in $\mathscr{R}^\times/\mathscr{O}_x^\times$, i.e. such that every definition function of D at x is either not defined at x or takes the value 0 at x.

Proposition 7. *The support of a divisor D on a variety X is a closed subset $\neq X$ of X, and $D = 0$ if and only if the support is empty.*

Proof. The latter claim is trivial. For the former, we prove that the set E of points $x \in X$ such that every definition function of D at x belongs to $\mathscr{O}_x^\times$ is a non-empty open subset; indeed, if we take a definition function g of D at x, then it is also a definition function of D in an open subset U that contains x. By hypothesis, if $x \in E$, then g is regular at x and $g(x) \neq 0$, and we can choose U such that g is regular and invertible on U, which proves that E is open. □

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Proposition 8. *If D is a divisor on a normal variety X, then* $\operatorname{supp} D$ *is a union of hypersurfaces (i.e. of closed subvarieties of codimension* 1*).*

Proof. If f is a function on a normal variety Y, we know that, if f is not defined at $x \in Y$, then x belongs to a variety of poles or zeros of f (i.e. to an irreducible component of the closure of the set of points $x \in Y$ such that $f(x) \in \{0, \infty\}$). So, if we take f to be a definition function for D in some open subset $U \subset X$, then $\operatorname{supp} D \cap U$ is the union of the pole and zero varieties of f in U, and we know that these varieties are of codimension 1 ([2, chapitre III]). □

Remark. If X is not normal, then the support of a divisor D on X is not necessarily of codimension 1. It is easy to define an affine variety X of dimension > 1 that is normal everywhere except at a single point x_0 (for example, the point (a, ab, b^2, b^3) in the four-dimensional space K^4). There exists a function u that is everywhere defined on X, and which is entire on the local ring of x_0, but which is not contained in this ring; by adding, if necessary, a constant, we can assume that x_0 is not a zero of u. There is then an open neighbourhood X' of x_0 such that the divisor of the function induced by u on X' has support equal to the single point x_0.

Suppose that X is a normal variety, and D is a divisor on X. Let S be a hypersurface of X. If f is a definition function of D at $x \in S$, then the order of f on S ([2]) does not depend on the choice of f, nor on $x \in S$. We denote this integer by $\operatorname{ord}_S D$. It is easy to see that $\mathrm{ord}_S D = 0$ if and only if $S \not\subset \mathrm{supp} D$. If we now take the formal combination $C = \sum_S (\mathrm{ord}_S D) \cdot S$, where S runs over the set of all hypersurfaces of X, then C is a cycle of codimension 1, and we call it the *associated cycle* of the divisor D.

Proposition 9. *Let X be a normal variety. The map that sends a divisor D to its associated cycle of codimension 1 is an injective homomorphism from the group of divisors on X to the group of cycles of codimension 1.*

Proof. The proof is trivial. □

Proposition 10. *If X is further a non-singular variety, then the homomorphism that sends a divisor to its associated cycle is bijective.* *p. 4-09*

Proof. It suffices to show that, for every hypersurface S, there exists a divisor D such that the cycle $1 \cdot S$ is the cycle associated to D. Since X is non-singular, for every $x \in X$, the local ring $\mathscr{O}_x$ is factorial [3]; thus, for every $x \in S$, S is defined by one single equation in a neighbourhood of x. So there exists a cover $\{U_i\}_{i=1,\ldots,p}$ of S by open subsets U_i of X, and, for each i, a regular function f_i on U_i that is non-zero outside of $U_i \cap S$ in U_i with $\mathrm{ord}_S f_i = 1$. It then follows that f_i/f_j is an invertible regular function in $U_i \cap U_j$. Now take the cover $\{U_i\}_{i=0,\ldots,p}$, where $U_0 = CS$, and take $f_0 = 1$. It is easy to see that the divisor D for which f_i is a definition function of D on U_i is such that the cycle associated to D is $1 \cdot S$. So the proposition is proven. □

Remark. Proposition 10 is not necessarily true if X is non-singular. For example, for the cone $xy - zw = 0$ in K^4, the cycle defined by $x = z = 0$ is not a cycle associated to any divisor.

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