The work of Koszul, II

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1949

This document is a translation into English of the following:

Cartan, H. "Les travaux de Koszul, II". *Séminaire Bourbaki* **1** (1952), Talk no. 8, pp. 45–52 numdam.org/book-part/SB_1948-1951__1__45_0.

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Contents

1 Reminder of notation

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$\mathfrak{a}$	Lie algebra over a field K	elements $x, y, \ldots$
$\mathfrak{a}'$	dual vector space of $\mathfrak{a}$	elements $x', y', \ldots$
$\Lambda(\mathfrak{a})$	exterior algebra of $\mathfrak{a}$	elements $\alpha, \beta, \ldots$
$\Lambda(\mathfrak{a}')$	exterior algebra of $\mathfrak{a}'$	elements $\alpha', \beta', \ldots$

1. The operator $\theta(x)$ is the *infinitesimal transformation defined by* $x \in \mathfrak{a}$ on $\Lambda(\mathfrak{a})$; from now on, we denote by $\theta^*(x)$ (instead of $\theta(x)$) the operator that is the transpose of $-\theta(x)$ acting on $\Lambda(\mathfrak{a}')$.

2. The operator $e(x)$ is the *(left) exterior product with* $x \in \mathfrak{a}$ on $\Lambda(\mathfrak{a})$; the operator $i(x)$ is the *interior product with* $x \in \mathfrak{a}$ on $\Lambda(\mathfrak{a}')$; $i(x)$ and $e(x)$ are the transpose of one another.
3. The operator ∂ is the *boundary* operator on $\Lambda(\mathfrak{a})$; the operator δ is the *coboundary* operator on $\Lambda(\mathfrak{a}')$; δ is the transpose of $-\partial$.

These operators satisfy the following relations:

$$\begin{aligned} \partial\partial &= 0, \\ \delta\delta &= 0, \\ \theta(x) &= e(x)\partial + \partial e(x), \\ \theta^*(x) &= i(x)\delta + \delta i(x), \\ \partial\theta(x) &= \theta(x)\partial, \\ \delta\theta^*(x) &= \theta^*(x)\delta. \end{aligned} \tag{1}$$

The *homology vector space* $\mathrm{H}(\mathfrak{a})$ is defined by ∂ acting on $\Lambda(\mathfrak{a})$; the *cohomology algebra* $\mathrm{H}(\mathfrak{a}')$ is defined by δ acting on $\Lambda(\mathfrak{a}')$. There is a multiplicative structure on $\mathrm{H}(\mathfrak{a}')$ because δ is an *antiderivation*. The vector spaces $\mathrm{H}(\mathfrak{a})$ and $\mathrm{H}(\mathfrak{a}')$ are canonically in duality.

2 Direct sum of two Lie algebras

We say that $\mathfrak{a}$ is the *direct sum* of two sub-algebras $\mathfrak{b}, \mathfrak{c} \subseteq \mathfrak{a}$, and we write $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{c}$, if

1. the vector space $\mathfrak{a}$ is the direct sum of the subspaces $\mathfrak{b}$ and $\mathfrak{c}$; and
2. $[x, y] = 0$ for $x \in \mathfrak{b}$ and $y \in \mathfrak{c}$.

If this is the case, then $\Gamma(\mathfrak{a})$ is canonically isomorphic to the tensor product $\Gamma(\mathfrak{b}) \otimes \Gamma(\mathfrak{c})$; if we transport the multiplicative structure of $\Gamma(\mathfrak{a})$ to this tensor product, then we see that *p. 46*

$$(\beta_1 \otimes \gamma_1) \cdot (\beta_2 \otimes \gamma_2) = (-1)^{pq}(\beta_1 \wedge \beta_2) \otimes (\gamma_1 \otimes \gamma_2)$$

(where $p = \deg \beta_2$ and $q = \deg \gamma_1$), which describes the notion of the *tensor product of graded algebras*. Finally, the boundary operator ∂ on $\Gamma(\mathfrak{b}) \otimes \Gamma(\mathfrak{c})$ satisfies

$$\partial(\beta \otimes \gamma) = (\partial\beta) \otimes \gamma + \overline{\beta} \otimes (\partial\gamma)$$

(where $\overline{\beta} = (-1)^p \beta$ for β of degree p). It thus follows that the vector space $\mathrm{H}(\mathfrak{a})$ can be identified with the tensor product $\mathrm{H}(\mathfrak{b}) \otimes \mathrm{H}(\mathfrak{c})$.

Similarly, $\Gamma(\mathfrak{a}')$ can be identified with the tensor product $\Gamma(\mathfrak{b}') \otimes \Gamma(\mathfrak{c}')$ of graded algebras. The duality between $\Gamma(\mathfrak{b}) \otimes \Gamma(\mathfrak{c})$ and $\Gamma(\mathfrak{b}') \otimes \Gamma(\mathfrak{c}')$ is defined by

$$\langle \beta \otimes \gamma, \beta' \otimes \gamma' \rangle = \langle \beta, \beta' \rangle \cdot \langle \gamma, \gamma' \rangle.$$

The operator δ on $\Gamma(\mathfrak{b}') \otimes \Gamma(\mathfrak{c}')$ satisfies

$$\delta(\beta' \otimes \gamma') = (\delta\beta') \otimes \gamma' + \overline{\beta'} \otimes (\delta\gamma'),$$

and the cohomology algebra $\mathrm{H}(\mathfrak{a}')$ can be identified with the tensor product $\mathrm{H}(\mathfrak{b}') \otimes \mathrm{H}(\mathfrak{c}')$ of *graded algebras*.

If $\mathfrak{b}$ and $\mathfrak{c}$ are the Lie algebras of *compact connected Lie groups* G_1 and G_2 (respectively), then $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{c}$ is the Lie algebra of the product group $G_1 \times G_2$; we recover the following theorem: *the cohomology algebra (with real coefficients) of $G_1 \times G_2$ is canonically identified with the tensor product of the cohomology algebras of G_1 and G_2.*

3 Semi-simple Lie algebras

Every time that we speak of semi-simplicity, the field K will be assumed to be of characteristic 0. We have the following equivalent characterisations of semi-simple Lie algebras

a. every square-zero ideal (i.e. giving rise to an invariant abelian subgroup) is the ideal $\{0\}$;

b. the quadratic form $\mathrm{tr}(\theta(x)\theta(y))$ is regular;

c. $\mathfrak{a}$ is the direct sum of *simple* algebras (i.e. algebras of dimension > 1 with only trivial ideals, namely $\{0\}$ and the whole algebra itself); such a decomposition is then unique. (In principal, determining $\mathrm{H}(\mathfrak{a})$ and $\mathrm{H}(\mathfrak{a}')$ for semi-simple $\mathfrak{a}$ thus reduces to determining the homology and cohomology of the simple algebras) ; and

d. every *representation* of $\mathfrak{a}$ in the endomorphism algebra of a vector space E (of finite dimension over K) is *completely reducible* (i.e. every invariant subspace admits an invariant complement).

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Every quotient algebra of a semi-simple algebra is semi-simple. We thus deduce: if $\mathfrak{a}$ is semi-simple, then $\mathfrak{a}$ is identical to the "derived" algebra $\mathfrak{a}^2$. In particular, a semi-simple algebra is *unimodular* (cf. §I).

For every Lie algebra $\mathfrak{a}$, the map $x \mapsto \theta(x)$ is a representation of $\mathfrak{a}$ in the endomorphism algebra of the vector space $\Gamma(\mathfrak{a})$; similarly, $x \mapsto \theta^*(\mathfrak{a})$ is a representation in the endomorphism algebra of $\Gamma(\mathfrak{a}')$. These representations vanish on the centre $\mathfrak{c}$ of $\mathfrak{a}$; we will be sure that they are completely reducible if the algebra $\mathfrak{a}/\mathfrak{c}$ is semi-simple, or, in other words, if $\mathfrak{a}$ is the direct sum of a semi-simple algebra and an abelian algebra. Such a Lie algebra is said to be *reductive*; we are particularly interested in the homology and cohomology of reductive Lie algebras. The Lie algebra of a compact group is always reductive.

4 Homology of reductive Lie algebras

In the representation $x \mapsto \theta(x)$ of $\mathfrak{a}$ in the endomorphism algebra of $\Gamma(\mathfrak{a})$, by (1), the $\theta(x)$ is a of a cycle is a boundary; similarly, the $\theta^*(x)$ of a cocycle is a coboundary.

Lemma 1. *Let $x \mapsto \tau(x)$ be a completely reducible representation of a Lie algebra $\mathfrak{a}$ in the endomorphism algebra of a finite-dimensional vector space E endowed with an operator d such that $dd = 0$; suppose that $\tau(x)d = d\tau(x)$, and that $\tau(x)$ sends cycles to boundaries. Let F be the subspace of "invariant" elements of E (i.e. elements α such that $\tau(x)\cdot\alpha = 0$ for all $x \in \mathfrak{a}$). Then F is stable under d, and the canonical homology from the homology group $\mathrm{H}(F)$ to the homology group $\mathrm{H}(E)$ is an* isomorphism *from the former to the latter.*

We can apply this to the representation $x \mapsto \theta(x)$ (resp. $x \mapsto \theta^*(x)$) of a *reductive* algebra $\mathfrak{a}$; then F becomes $\mathfrak{J}$ (resp. $\mathfrak{J}'$), the sub-algebra of *invariant* chains (resp. *invariant* cochains). But ∂ is zero for every element of $\mathfrak{J}$, and δ is zero for every element of $\mathfrak{J}'$, thanks to the formulas

$$2\delta = \sum_k e(x'_k)\theta(x_k)$$
$$2\partial = \sum_k i(x'_k)^*\theta(x_k)$$

(where (x_k) and (x'_k) are dual bases; see §I). So $\mathrm{H}(\mathfrak{J})$ can be identified with $\mathfrak{J}$, and $\mathrm{H}(\mathfrak{J}')$ with $\mathfrak{J}'$, and the lemma shows that $\mathfrak{J} \to \mathrm{H}(\mathfrak{a})$ and $\mathfrak{J}' \to \mathrm{H}(\mathfrak{a})$ are *onto isomorphisms* (i.e there is one and only one invariant chain in each homology class; id. for cochains and cohomology). Furthermore, $\mathfrak{J}' \to \mathrm{H}(\mathfrak{a}')$ is an isomorphism for the *algebra* structure; we can not say the same of $\mathfrak{J} \to \mathrm{H}(\mathfrak{a})$, since $\mathrm{H}(\mathfrak{a})$ has no multiplicative structure, but the identification of $\mathrm{H}(\mathfrak{a})$ with $\mathfrak{J}$, which does have an algebra structure, precisely defines a multiplicative structure on $\mathrm{H}(\mathfrak{a})$, and allows us to speak about the *homology algebra* of a reductive Lie algebra. We will prove that, if $\mathfrak{a}$ is the Lie algebra of a compact connected group, then the multiplicative structure of $\mathrm{H}(\mathfrak{a})$ is precisely that which is defined by the "Pontrjagin product". *p. 48*

Remark. On any reductive $\mathfrak{a}$, there exists at least one *regular* invariant quadratic form; it defines an isomorphism from the algebra $\Gamma(\mathfrak{a})$ to the algebra $\Gamma(\mathfrak{a})$, and from $\mathfrak{J}$ to $\mathfrak{J}'$; thus $\mathrm{H}(\mathfrak{a})$ and $\mathrm{H}(\mathfrak{a}')$ are (non-canonically) *isomorphic algebras*.

5 The first three Betti numbers of a semi-simple Lie algebra

A preliminary remark: for α' to be an *invariant* cochain, it is necessary and sufficient that it be a cocycle, *and* that the $i(x)\cdot\alpha'$ be cocycles for *all* $x \in \mathfrak{a}$ (by (1)).

The first Betti number is zero. The equation

$$\langle x \wedge y, \delta x' \rangle = -\langle [x,y], x' \rangle$$

shows that, if x' is a cocycle, then x' is orthogonal to the derived algebra $\mathfrak{a}^2$, and is thus zero if $\mathfrak{a}$ is semi-simple.

The second Betti number is zero. Every invariant cochain α' of degree 2 is zero, since $i(x)\cdot\alpha'$ is a cocycle of degree 1 for all x, and is thus zero.

The third Betti number is equal to the dimension of the vector space of invariant quadratic forms. In particular, it is always $\geqslant 1$. An invariant quadratic form is a *symmetric* bilinear form $f(x,y)$ such that $f(\theta(z)\cdot x, y) + f(x, \theta(z)\cdot y) = 0$ for all $z \in \mathfrak{a}$. We define, as follows, an isomorphism from the vector space $(\mathfrak{J}')^{(3)}$ of invariant cochains of degree 3 to the space of invariant quadratic forms: let α' be invariant and of degree 3; for every $x \in \mathfrak{a}$, we know that $i(x)\cdot\alpha'$ is a cocycle of degree 2, and thus cohomologous to 0; there thus exists a unique cochain x' of degree 1 such that $\delta x' = i(x)\cdot\alpha'$; the map $x \mapsto x'$ from $\mathfrak{a}$ to $\mathfrak{a}'$ defines a bilinear form $f(x,y) = \langle y, x' \rangle$. We have *p. 49*

$$f(x,[y,z]) = -\langle x \wedge y \wedge z, \alpha' \rangle, \tag{2}$$

and f is symmetric and invariant. Conversely, if f is a symmetric invariant bilinear form, then there exists a unique cochain α' such that (2) holds, and this cochain is invariant. QED.

Remark. This converse holds even without the semi-simplicity assumption. In particular, the invariant symmetric bilinear form $\operatorname{tr}\theta(x)\theta(y)$ defines an invariant cochain of degree 3, called the *Cartan cochain*.

Corollary. *If* $\mathfrak{a}$ *is a Lie algebra such that the* uniqueness *(up to a scalar multiple) of an invariant quadratic form is assured, then the third Betti number of* $\mathfrak{a}$ *is equal to* 1. *This is notably the case for the algebra of a compact connected Lie group, or for a simple Lie algebra over an algebraically closed field.*

6 Homomorphisms from one Lie algebra to another

Let $\mathfrak{b}$ and $\mathfrak{a}$ be arbitrary Lie algebras (over the same field K), and let φ be a homomorphism from $\mathfrak{b}$ to $\mathfrak{a}$; then φ extends to a homomorphism, again denoted by φ, from the *algebra* $\Gamma(\mathfrak{b})$ to the *algebra* $\Gamma(\mathfrak{a})$. The transpose φ^* (a homomorphism from $\mathfrak{a}'$ to $\mathfrak{b}'$) can be extended to a homomorphism, again denoted by φ^*, from the algebra $\Gamma(\mathfrak{a}')$ to the algebra $\Gamma(\mathfrak{b}')$; furthermore, the extensions of φ and φ^* are *transpose to one another*. We have $\partial\varphi = \varphi\partial$ and $\delta\varphi^* = \varphi^*\delta$, whence a homomorphism $\widetilde{\varphi}$ from $\mathrm{H}(\mathfrak{b})$ to $\mathrm{H}(\mathfrak{a})$, and a homomorphism $\widetilde{\varphi}^*$ from $\mathrm{H}^(\mathfrak{a}')$ to $\mathrm{H}(\mathfrak{b}')$, with $\widetilde{\varphi}^*$ being the transpose of $\widetilde{\varphi}$. The homomorphism $\widetilde{\varphi}^*$ is also a homomorphism for the multiplicative structures.

Theorem 1. *If the algebras* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are* reductive*, then* $\widetilde{\varphi}$ *is a homomorphism for the multiplicative structures of* $\mathrm{H}(\mathfrak{b})$ *and* $\mathrm{H}(\mathfrak{a})$*.*

This can be proved by using:

Lemma 2. *Let* $\mathfrak{a}$ *be an arbitrary Lie algebra; then, for any chains* α *and* β *of* $\Gamma(\mathfrak{a})$*, the chain*

$$\partial(\alpha\wedge\beta) + (\partial\alpha)\wedge\beta - \overline{\alpha}\wedge(\partial\beta)$$

is orthogonal to the invariant cochains.

With this, we consider a reductive Lie algebra, and we will show that, if α and$\overline{\alpha}\wedge\partial\gamma$ are cycles, then $\overline{\alpha}\wedge\partial\gamma$ is a boundary: indeed, by Lemma 2, $\partial(\alpha\wedge\gamma) - \overline{\alpha}\wedge\partial\gamma$ is orthogonal *p. 50* to the invariant cochains, and, since it is a cycle, it is orthogonal to the coboundaries; it is thus orthogonal to all the cocycles, and thus is a boundary; so $\overline{\alpha}\wedge\partial\gamma$ is indeed a boundary. Now let α and β be invariant cycles, and α_1 and β_1 cycles that are homologous to α and β (respectively); we will show that, if $\alpha_1\wedge\beta_1$ is a cycle, then this cycle is homologous to $\alpha\wedge\beta$: we have

$$\alpha\wedge\beta - \alpha_1\wedge\beta_1 = \alpha\wedge(\beta-\beta_1) + (\alpha-\alpha_1)\wedge\beta,$$

and, by what we have just shown, both of the terms on the right-hand side are boundaries.

From this it follows that, in a reductive Lie algebra, if α_1 and α_2 are cycles such that $\alpha_1\wedge\beta_1$ is a cycle, then the homology class of $\alpha_1\wedge\beta_1$ is the product of the homology classes of α_1 and β_1.

The proof of Theorem 1 goes as follows: β_1 and β_2 are *invariant* chains of $\Gamma(\mathfrak{b})$, and so $\varphi(\beta_1)$, $\varphi(\beta_2)$, and $\varphi(\beta_1\wedge\beta_2)$ are all cycles in $\Gamma(\mathfrak{a})$, and so the class of $\varphi(\beta_1\wedge\beta_2) = \varphi(\beta_1)\wedge\varphi(\beta_2)$ is the product of the classes of $\varphi(\beta_1)$ and $\varphi(\beta_2)$.

7 The Hopf–Samelson theorem

This theorem is proven for reductive Lie algebras, using Theorem 1 above. First, let $\mathfrak{a}$ be an arbitrary Lie algebra, and φ the "diagonal map" from $\mathfrak{a}$ to $\mathfrak{a}\oplus\mathfrak{a}\subset\Gamma(\mathfrak{a})\otimes\Gamma(\mathfrak{a})$:

$$\varphi(x) = x\oplus x = x\otimes 1 + 1\otimes x.$$

From this, we obtain a homomorphism φ from $\Gamma(\mathfrak{a})$ to $\Gamma(\mathfrak{a})\otimes\Gamma(\mathfrak{a})$, such that

$$\varphi(\alpha) = \alpha\times 1 + \ldots + 1\otimes\alpha.$$

The transpose map φ^* from $\Gamma(\mathfrak{a}')\otimes\Gamma(\mathfrak{a}')$ to $\Gamma(\mathfrak{a}')$ is an *algebra* homomorphism (§II.6); it easily follows that

$$\varphi^*(\alpha'\otimes\beta') = \alpha'\wedge\beta'.$$

The homomorphism $\widetilde{\varphi}$ from $\mathrm{H}(\mathfrak{a})$ to $\mathrm{H}(\mathfrak{a}) \otimes \mathrm{H}(\mathfrak{a})$ satisfies

$$\widetilde{\varphi}(\widetilde{\alpha}) = \widetilde{\alpha} \otimes 1 + \ldots + 1 \otimes \widetilde{\alpha}$$

where $\widetilde{\alpha}$ denotes a homology class. The transpose homomorphism $\widetilde{\varphi}^*$ is an *algebra homomorphism*, and satisfies

$$\widetilde{\varphi}^*(\widetilde{\alpha}' \otimes \widetilde{\beta}') = \widetilde{\alpha}' \cdot \widetilde{\beta}'$$

(where the product $\cdot$ is in $\mathrm{H}(\mathfrak{a}')$).

If $\mathfrak{a}$ is now a *reductive* algebra, then the homomorphism $\widetilde{\varphi}$ is also a homomorphism *for the multiplicative structures* of $\mathrm{H}(\mathfrak{a})$ and $\mathrm{H}(\mathfrak{a}) \otimes \mathrm{H}(\mathfrak{a})$. We thus find ourselves in the following algebraic situation: *p. 51*

K is a field of characteristic 0; A is a graded algebra of finite rank over K whose multiplication law satisfies the usual commutativity and anticommutativity conditions for an exterior algebra; A' is a graded algebra that is (non-canonically) isomorphic to A, in canonical duality with A; the degree 0 elements of A (resp. of A') are given by multiples of the unit, and can be identified with the elements of K. The duality between $A \otimes A$ and $A' \otimes A'$ is defined by $\langle \alpha \otimes \beta, \alpha' \otimes \beta' \rangle = \langle \alpha, \alpha' \rangle \cdot \langle \beta, \beta' \rangle$, and the transpose of the canonical homomorphism $\widetilde{\varphi}^*$ from $A' \otimes A'$ to A' (such that $\widetilde{\varphi}^*(\alpha' \otimes \beta') = \alpha' \cdot \beta'$) is $\widetilde{\varphi}$, which is a homomorphism from A to $A \otimes A$ *that respects the multiplicative structures*. In such a situation, we say that the homogeneous elements of degree $\geqslant 1$ of A (resp. of A') that are orthogonal to the products $\alpha' \cdot \beta'$ with α' and β' of degree $\geqslant 1$ (resp. to the products $\alpha \cdot \beta$ etc.) are *primitive*. We can then prove the following theorem of algebra:

The Hopf–Samelson theorem. *The* primitive *elements of are* odd degree*; they generate a vector subspace P of A (resp. P' of A'); the canonical map $P \to A$ (resp. $P' \to A'$) can be extended to give an* isomorphism from the exterior algebra $\Gamma(P)$ onto A *(resp. an isomorphism from* $\Gamma(P')$ *onto A'); when restricted to P and P', the duality between A and A' defines a duality between P and P'; this duality can be extended in the usual way to give a duality between* $\Gamma(P)$ *and* $\Gamma(P')$ *that, by identifying* $\Gamma(P)$ *with A, and* $\Gamma(P')$ *with A', recovers the existing duality between A and A'.*

The above statement thus applies if we take A to be the homology algebra $\mathrm{H}(\mathfrak{a})$ of a reductive Lie algebra, and A' to then be the cohomology algebra $\mathrm{H}(\mathfrak{a}')$.

Remark. In the case where $\mathfrak{a}$ is the Lie algebra of a compact connected group, the dimension of the space P of primitive elements is equal to the *rank* of the group, that is, to the maximal dimension of the abelian sub-algebras of $\mathfrak{a}$. We still do not have an algebraic proof of this fact, which would prove this statement for a reductive Lie algebra.

8 Homomorphisms from a reductive algebra to a reductive algebra

Let $\mathfrak{a}$ and $\mathfrak{b}$ be reductive algebras, and φ a homomorphism from $\mathfrak{b}$ to $\mathfrak{a}$; we keep the notation form §II.6. Then every *primitive* element of $\mathrm{H}(\mathfrak{b})$ is sent, by $\widetilde{\varphi}$, to a *primitive* element of $\mathrm{H}(\mathfrak{a})$; every *primitive* element of $\mathrm{H}(\mathfrak{a}')$ is sent, by $\widetilde{\varphi}^*$, to a *primitive* element of $\mathrm{H}(\mathfrak{b}')$. Taking Theorem 1 into account (from §II.6), we thus deduce: *p. 52*

Theorem 2. (cf. Samelson). *The ideal of zeros of $\widetilde{\varphi}$ is the* ideal *of* $\mathrm{H}(\mathfrak{b})$ *generated by the* primitive *elements of* $\mathrm{H}(\mathfrak{b})$ *whose image under $\widetilde{\varphi}$ is zero; the sub-algebra given by the image of*

$\widetilde{\varphi}$ is the sub-algebra *of* $\mathrm{H}(\mathfrak{a})$ *generated by the unit and by the* primitive *elements of* $\mathrm{H}(\mathfrak{a})$ *that belong to the image of* $\widetilde{\varphi}$*. The analogous properties hold for cohomology algebras.*

Corollary. *Let n denote the dimension of* $\mathfrak{b}$*. For $\widetilde{\varphi}$ to be* bijective*, it suffices that $\widetilde{\varphi}(\omega) \neq 0$, where ω is the generating element of* $\mathrm{H}_n(\mathfrak{b})$*.*

The next talk will be dedicated to the study of *relative* homology and cohomology: given a sub-algebra $\mathfrak{b}$ of $\mathfrak{a}$, we define the homology (resp. cohomology) of the "homogeneous space" $\mathfrak{a}/\mathfrak{b}$, and we study the relations between the homology (resp. cohomology) of $\mathfrak{a}$, of $\mathfrak{b}$, and of $\mathfrak{a}/\mathfrak{b}$.