The work of Koszul, III

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Contents

We keep the notation from the previous talk. However, we now write $\Gamma^*(\mathfrak{a})$ (instead of $\Gamma(\mathfrak{a}'))$ to denote the exterior algebra of the dual of $\mathfrak{a}$, which is in duality with $\Gamma(\mathfrak{a})$ (the exterior algebra of $\mathfrak{a}$). We denote by $\mathrm{H}^*(\mathfrak{a})$ (instead of $\mathrm{H}(\mathfrak{a}')$) the *cohomology algebra* of $\mathfrak{a}$, and by $\mathrm{H}^p(\mathfrak{a})$ the subspace of homogeneous elements of degree p of $\mathrm{H}^*(\mathfrak{a})$. We keep the notation $\mathrm{H}(\mathfrak{a})$ for the homology *space* of $\mathfrak{a}$, and $\mathrm{H}_p(\mathfrak{a})$ for the subspace of elements of degree p of $\mathrm{H}(\mathfrak{a})$. If $\mathfrak{a}$ is a *reductive* Lie algebra, then, as we have seen, $\mathrm{H}(\mathfrak{a})$ is endowed with a *multiplicative structure*: we denote by $\mathrm{H}_*(\mathfrak{a})$ the *homology algebra* of $\mathfrak{a}$ (i.e. the space $\mathrm{H}(\mathfrak{a})$ endowed with this multiplicative structure).

Recall that a Lie algebra $\mathfrak{a}$ (over a field K) is said to be *reductive* if the representation $x \mapsto \theta(x)$ of $\mathfrak{a}$ in the endomorphism algebra of the vector space $\Gamma(\mathfrak{a})$ is completely reducible; such an algebra $\mathfrak{a}$ is the direct sum of a semi-simple Lie algebra and an abelian algebra, and the converse is true if K is of *characteristic* 0. More generally, let $\mathfrak{b}$ be a sub-algebra of a Lie algebra $\mathfrak{a}$; we say that $\mathfrak{b}$ is *reductive in* $\mathfrak{a}$ if the representation $x \mapsto \theta(x)$ of $\mathfrak{b}$ in the endomorphism algebra of $\Gamma(\mathfrak{a})$ is completely reducible; the $\mathfrak{b}$ is also reductive (i.e. reductive in itself).

If $\mathfrak{a}$ is a Lie algebra of a *compact* group (an algebra over the field of reals), then every sub-algebra of $\mathfrak{a}$ is reductive in $\mathfrak{a}$.

1 Relative chains and cochains

Let $\mathfrak{b}$ be a sub-algebra of a Lie algebra $\mathfrak{a}$; this gives a bijective homomorphism $\varphi\colon \Gamma(\mathfrak{b}) \to \Gamma(\mathfrak{a})$, and its transpose $\varphi^*: \Gamma^*(\mathfrak{a}) \to \Gamma^*(\mathfrak{b})$ (which is *onto*). A subspace of $\Gamma(\mathfrak{a})$ (resp. of $\Gamma^*(\mathfrak{a})$) is said to be $\mathfrak{b}-stable$ if it is stable under the endomorphisms $\theta(x)$ (resp. $\theta^*(x)$) for all $x \in \mathfrak{b}$. We denote by $I(\mathfrak{a},\mathfrak{b})$ the subspace of chains of $\mathfrak{a}$ that are *invariant under* $\mathfrak{b}$, that is, the $\alpha \in \Gamma(\mathfrak{a})$ such that $\theta(x)\cdot\alpha = 0$ for all $x \in \mathfrak{b}$. There is the analogous definition of the subspace $I^*(\mathfrak{a},\mathfrak{b})$ of $\mathfrak{b}$-invariant cochains of $\mathfrak{a}$. *p. 72*

Let $N(\mathfrak{a},\mathfrak{b})$ be the ideal of $\Gamma(\mathfrak{a})$ generated by $\varphi(\mathfrak{b})$; the subspace of $\Gamma^*(\mathfrak{a})$ that is orthogonal to $N(\mathfrak{a},\mathfrak{b})$ is the *sub-algebra* $N^{(}(\mathfrak{a},\mathfrak{b})$ generated by the unit and the degree 1 cochains that are orthogonal to $\mathfrak{b}$. We write $N^p(\mathfrak{a},\mathfrak{b})$ to mean the subspace of degree p elements of $N^*(\mathfrak{a},\mathfrak{b})$. The subspaces $N(\mathfrak{a},\mathfrak{b})$ and $N^*(\mathfrak{a},\mathfrak{b})$ are $\mathfrak{b}$-stable.

We first study the particular case where $\mathfrak{b}$ is an *invariant* sub-algebra (an ideal of the Lie algebra $\mathfrak{a}$), that is, the case where $\mathfrak{b}$ is $\mathfrak{a}$-stable, in which case we have the Lie algebra $\mathfrak{a}/\mathfrak{b}$. Then $N(\mathfrak{a},\mathfrak{b})$ is *stable under* ∂, and $\Gamma(\mathfrak{a})/N(\mathfrak{a},\mathfrak{b})$, endowed with the operator induced by ∂ by passing to the quotient, can be identified with $\Gamma(\mathfrak{a}/\mathfrak{b})$; by duality, $N^{(}(\mathfrak{a},\mathfrak{b})$ endowed with δ (under which it is stable) can be identified with $\Gamma^*(\mathfrak{a}/\mathfrak{b})$, and is contained inside $I^*(\mathfrak{a},\mathfrak{b})$.

In the general case of an arbitrary sub-algebra $\mathfrak{b}$, the subspace $N(\mathfrak{a},\mathfrak{b}) + \partial N(\mathfrak{a},\mathfrak{b})$ is stable under ∂, and is generated by $N(\mathfrak{a},\mathfrak{b})$ and by the $\theta(x)\cdot\alpha$ (where $\alpha \in \Gamma(\mathfrak{a})$ and $x \in \mathfrak{b}$); then ∂ acts on the quotient $L(\mathfrak{a},\mathfrak{b})$ of $\Gamma(\mathfrak{a})$ by $N(\mathfrak{a},\mathfrak{b}) + \partial N(\mathfrak{a},\mathfrak{b})$, and we call this quotient the *space of relative chains*. By duality, the intersection $N^*(\mathfrak{a},\mathfrak{b}) \cap I^*(\mathfrak{a},\mathfrak{b}) = L^*(\mathfrak{a},\mathfrak{b})$ consists of the cochains α' of $N^*(\mathfrak{a},\mathfrak{b})$ such that $\delta\alpha' \in N^*(\mathfrak{a},\mathfrak{b})$; this is a *sub-algebra* of $\Gamma^*(\mathfrak{a})$, called the *algebra of relative cochains*.

Endowing $L(\mathfrak{a},\mathfrak{b})$ with ∂ defines a *relative homology space*, denoted by $\mathrm{H}(\mathfrak{a},\mathfrak{b})$; it is graded. The algebra $L^*(\mathfrak{a},\mathfrak{b})$ endowed with δ defines a *relative cohomology algebra*, denoted by $\mathrm{H}^*(\mathfrak{a},\mathfrak{b})$; this is a graded algebra. In the particular case where $\mathfrak{b}$ is an invariant sub-algebra of $\mathfrak{a}$, we can identify $\mathrm{H}(\mathfrak{a},\mathfrak{b})$ (resp. $\mathrm{H}^*(\mathfrak{a},\mathfrak{b})$) with $\mathrm{H}(\mathfrak{a}/\mathfrak{b})$ (resp. $\mathrm{H}^*(\mathfrak{a}/\mathfrak{b})$).

In the case where $\mathfrak{a}$ is the Lie algebra of a *compact connected group* G, and $\mathfrak{b}$ is the sub-algebra of a *closed subgroup* U, then $L^*(\mathfrak{a},\mathfrak{b})$ can be identified with the algebra of exterior differential forms of the *homogeneous space* $W = G/U$ that are invariant under G, and $H^*(\mathfrak{a},\mathfrak{b})$ is then the cohomology algebra of the (compact) topological space W. The following results (some of which are new) explain the relations between the cohomology algebras of the spaces G, U, and $W = G/U$. We note that G is the fibre bundle with fibre U and base W.

2 Canonical homomorphisms

We have

$$\Gamma(\mathfrak{b}) \xrightarrow{\varphi} \Gamma(\mathfrak{a}) \xrightarrow{\pi} L(\mathfrak{a},\mathfrak{b})$$

$$L^*(\mathfrak{a},\mathfrak{b}) \xrightarrow{\pi^*} \Gamma^*(\mathfrak{a}) \xrightarrow{\varphi^*} \Gamma^*(\mathfrak{b}).$$

These define *p. 73*

$$\mathrm{H}(\mathfrak{b}) \xrightarrow{\widetilde{\varphi}} \mathrm{H}(\mathfrak{a}) \xrightarrow{\widetilde{\pi}} \mathrm{H}(\mathfrak{a},\mathfrak{b})$$

$$\mathrm{H}^*(\mathfrak{a},\mathfrak{b}) \xrightarrow{\widetilde{\pi}^*} \mathrm{H}^*(\mathfrak{a}) \xrightarrow{\widetilde{\varphi}^*} \mathrm{H}^*(\mathfrak{b}).$$

The homomorphisms π^*, φ^*, $\widetilde{\pi}^*$, and $\widetilde{\varphi}^*$ are *algebra homomorphisms*. The image of $\widetilde{\varphi}$ is contained inside the kernel of $\widetilde{\pi}$; the image of $\widetilde{\varphi}^*$ is contained inside the kernel of $\widetilde{\pi}^*$.

If $\mathfrak{a}$ is a *reductive* algebra, then the kernel of $\mathrm{H}(\mathfrak{a}) \to \mathrm{H}(\mathfrak{a},\mathfrak{b})$ is a two-sided *ideal* of the *algebra* $\mathrm{H}_*(\mathfrak{a})$. If, furthermore, K is of characteristic 0, then the Hopf theorem applies to $\mathrm{H}^*(\mathfrak{a})$, which can be identified with the exterior algebra of the subspace of its *primitive* elements; then the image of $\mathrm{H}^*(\mathfrak{a},\mathfrak{b}) \to \mathrm{H}^*(\mathfrak{a})$ is a sub-algebra of $\mathrm{H}^*(\mathfrak{a})$ that is *generated by the unit and by the primitive elements of* $\mathrm{H}^*(\mathfrak{a})$.

If $\mathfrak{b}$ is a *reductive sub-algebra of* $\mathfrak{a}$, then we have a homology algebra $\mathrm{H}_*(\mathfrak{b})$, and the kernel of $\mathrm{H}(\mathfrak{b}) \to \mathrm{H}(\mathfrak{a})$ is a two-sided *ideal* of $\mathrm{H}_*(\mathfrak{b})$. If, furthermore, K is of characteristic 0, then the image of $\mathrm{H}^*(\mathfrak{a}) \to \mathrm{H}^*(\mathfrak{b})$ is a sub-algebra of $\mathrm{H}^*(\mathfrak{b})$ that is *generated by the unit and by the primitive elements of* $\mathrm{H}^*(\mathfrak{b})$.

3 Poincaré duality for relative homology and cohomology

Let n (resp. m) be the dimension of $\mathfrak{a}$ (resp. $\mathfrak{b}$). Let ω be the image in $\Gamma(\mathfrak{a})$ of the m-dimensional chain of $\Gamma(\mathfrak{b})$ (defined up to a constant factor). Then $N(\mathfrak{a},\mathfrak{b})$ is the kernel of the endomorphism $\alpha \mapsto \omega \wedge \alpha$ of $\Gamma(\mathfrak{a})$. Let $\tau \in \Gamma^{n-m}(\mathfrak{a})$ be such that $\omega \wedge \tau \neq 0$; then, in $\Gamma(\mathfrak{a})/N(\mathfrak{a},\mathfrak{b})$, every element of degree $>(n-m)$ is zero, and the elements of degree $(n-m)$ are proportional to the class $\dot{\tau}$ of τ. So $\alpha' \mapsto i(\alpha')\cdot\tau$ defines an *isomorphism* from $N^*(\mathfrak{a},\mathfrak{b})$ *onto* $\Gamma(\mathfrak{a})/N(\mathfrak{a},\mathfrak{b})$ which depends only on $\dot{\tau}$.

From now on, suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are *unimodular*; then $\dot{\tau}$ is a $\mathfrak{b}$-invariant element of $\Gamma(\mathfrak{a})/N(\mathfrak{a},\mathfrak{b})$, and so $\alpha' \mapsto i(\alpha')\cdot\lambda$ defines an isomorphism from $N^*(\mathfrak{a},\mathfrak{b}) \cap I^*(\mathfrak{a},\mathfrak{b}) = L^*(\mathfrak{a},\mathfrak{b})$ onto the subspace of $\mathfrak{b}$-invariant elements of $\Gamma(\mathfrak{a})/N(\mathfrak{a},\mathfrak{b})$.

If, furthermore, $\mathfrak{b}$ is *reductive in* $\mathfrak{a}$, then this subspace can be identified with $L(\mathfrak{a},\mathfrak{b}) = \Gamma(\mathfrak{a})/N(\mathfrak{a},\mathfrak{b}) + \partial N(\mathfrak{a},\mathfrak{b})$. This gives an isomorphism f from $L^*(\mathfrak{a},\mathfrak{b})$ onto $L(\mathfrak{a},\mathfrak{b})$. Furthermore,

$$\partial i(\alpha')\cdot\tau = i(\delta\overline{\alpha}')\cdot\tau + i(\overline{\alpha}')\cdot\partial\tau.$$

But the fact that $\mathfrak{b}$ is reductive in $\mathfrak{a}$ implies that $\partial\tau \in N + \partial N$, and since $i(\overline{\alpha}')\cdot(N+\partial N) \subset N+\partial N$, we have that $\partial f(\alpha') = f(\delta\overline{\alpha}')$. By passing to quotients, f thus defines an *isomorphism from* $\mathrm{H}(\mathfrak{a},\mathfrak{b})$ *onto* $\mathrm{H}^*(\mathfrak{a},\mathfrak{b})$ that sends degree p elements to degree $(n-m-p)$ elements. We thus *p. 74*
deduce that the "relative" Betti numbers are equal for dimensions p and $(n-m-p)$ ("Poincaré duality"), and that *every non-zero ideal of* $\mathrm{H}^*(\mathfrak{a},\mathfrak{b})$ *contains* $\mathrm{H}^{n-m}(\mathfrak{a},\mathfrak{b})$.

4 Sub-algebras that are not homologous to zero

From now on, we also suppose that the sub-algebra $\mathfrak{b}$ of $\mathfrak{a}$ is *reductive in* $\mathfrak{a}$. Let m be the dimension of $\mathfrak{b}$.

For $\mathrm{H}(\mathfrak{b}) \xrightarrow{\widetilde{\varphi}} \mathrm{H}(\mathfrak{a})$ to be *bijective*, it *suffices* for $\mathrm{H}_m(\mathfrak{b})$ (which consists of multiples of a single element) to not be contained in the kernel of $\widetilde{\varphi}$ (since every two-sided ideal of $\mathrm{H}_*(\mathfrak{b})$ that does not contain $\mathrm{H}_m(\mathfrak{b})$ is zero). We then say that $\mathfrak{b}$ is *not homologous to zero in* $\mathfrak{a}$.

If $\mathfrak{b}$ is not homologous to zero in $\mathfrak{a}$, then $\mathrm{H}^*(\mathfrak{a},\mathfrak{b}) \to \mathrm{H}^*(\mathfrak{a})$ is *bijective* (and the converse is also true). So, if $\mathfrak{a}$ is further assumed to be reductive, and the base field to be of characteristic 0, then $\mathrm{H}^*(\mathfrak{a},\mathfrak{b})$ has a subspace whose homogeneous components are of *odd degree*, and that can be identified with the *exterior algebra of this subspace*.

If $\mathfrak{b}$ is not homologous to zero in $\mathfrak{a}$, then $\mathrm{H}^*(\mathfrak{a}) \to \mathrm{H}^*(\mathfrak{b})$ is evidently *onto*. If, furthermore, the field is of characteristic 0, then we can define an isomorphism from the algebra $\mathrm{H}^*(\mathfrak{b})$ to the algebra $\mathrm{H}^*(\mathfrak{a})$, such that, by composing with the canonical map $\mathrm{H}^*(\mathfrak{a}) \to \mathrm{H}^*(\mathfrak{b})$, we obtain the identity automorphism of $\mathrm{H}^*(\mathfrak{b})$. This map $\mathrm{H}^*(\mathfrak{b}) \to \mathrm{H}^*(\mathfrak{a})$, along with the canonical map $\mathrm{H}^*(\mathfrak{a},\mathfrak{b}) \to \mathrm{H}^*(\mathfrak{a})$, define a *homomorphism of (graded) algebras*

$$\mathrm{H}^*(\mathfrak{b}) \otimes \mathrm{H}^*(\mathfrak{a},\mathfrak{b}) \to \mathrm{H}^*(\mathfrak{a}),$$

and we can show that this is an *onto isomorphism* ("Samelson's theorem").

The case of *sub-algebras that are homologous to zero* will not be examined here (see instead the Koszul's thesis: *Bull. Soc. Math. France* **78** (1950), 65–127).