
On modifications and exceptional analytic sets

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Translator's note

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The term “modification” first appeared in a 1951 publication [1] by H. Behnke and K. Stein. The authors used it to refer to a process that allows a given complex space to be modified. If X is a complex space, and $N \subset X$ a low-dimensional analytic set, then N is replaced by another set N' such that the complex structure on $X \setminus N$ can be extended to the entire space $X' = (X \setminus N) \cup N'$. The newly obtained complex space X' is then called a *modification* of X .

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As already demonstrated in [1], modifications can be very pathological. The interest therefore turned towards special classes of modifications. In [12], H. Hopf considered so-called “ σ -processes” on n -dimensional complex manifolds M . These modifications made it possible to replace any point $x \in M$ with a complex projective space $\mathbb{P}^{n-1}$ of dimension $n - 1$. The result is a new singularity-free complex manifold M' . There are more general modifications that modify the manifold M at only one point $x \in M$, but the space thus obtained can then contain singular points, i.e. is just a complex space.

This present work deals with the following question. Let X be a complex space, and $A \subset X$ a complex-analytic set. Then when does there exist a modification Y of X where A is replaced by a point y , and such that $X \setminus A = Y \setminus y$?

If such a Y exists, then A is said to be an *exceptional analytic set* in X , and we say that A can be “*collapsed*” to a point.

In general, such a Y does not exist. If X is a complex space, and $A \subset X$ is a compact connected analytic set, then, from a set-theoretic point of view, A can of course always be replaced by a point y_0 . Then $Y = (X \setminus A) \cup y_0$ has a canonical topological structure, $Y \setminus y_0 = X \setminus A$ has a complex structure $\mathfrak{S}$, and the identity $X \setminus A \rightarrow X \setminus y_0$ can be extended to a continuous map $\lambda: X \rightarrow Y$. Then λ maps $X \setminus A$ topologically (and even biholomorphically) to Y , and sends A to y_0 . If A can now be collapsed to a point, then $\mathfrak{S}$ can be extended to the entire space Y , and λ becomes a holomorphic map $X \rightarrow Y$.

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We now give an overview of the present work. In §1 we study the concepts of *pseudoconvexity* and *holomorphic convexity* on complex spaces. The reduction theory of Remmert

then leads, in §2, to the first general theorem concerning exceptional analytic sets $A \subset X$. In order to simplify the somewhat strong assumption in this theorem, we consider, in §3, a coherent analytic sheaf $\mathfrak{m}$ of germs of holomorphic functions that vanish on A , so that A is exactly the zero set of $\mathfrak{m}$. Using $\mathfrak{m}$, we then endow A with a normal bundle $N_{\mathfrak{m}}$. The structure of $N_{\mathfrak{m}}$ is then critical: A is exceptional if N is weakly negative. We use the word “negative” here in the sense of Kodaira’s definition; our result shows that it can be defined in a purely algebraic way in the world of algebraic geometry. Also in §3, we obtain simple criteria for positive (negative) line bundles and characterising projective algebraic spaces. The well-known theorem of Kodaira (that every Hodge manifold X is projective algebraic) is generalised to the case where X is a normal complex space. Then §4 deals with the complex structure of neighbourhoods of analytic sets $A \subset X$, which can be collapsed to a point. The main result of this section is that the neighbourhoods of (special) exceptional analytic sets $A \subset X$ and $A' \subset X'$ are analytically equivalent if they are equivalent in a formal sense. This means that the complex structure can be “calculated,” which makes it possible to solve one of Hirzebruch’s problems [11], and to transfer the propositions of Enriques and Kodaira from algebraic geometry to complex analysis.

— It should also be mentioned that, using the main results of §4, we construct a complex space X with the following properties:

1. X is connected, compact, and of dimension 2;
2. X is normal, and has only one non-regular points;
3. there exist two analytically and algebraically independent meromorphic functions on X ; and
4. X is not an algebraic variety (neither in the projective sense nor the more general sense of Weil).¹

In contrast, as is well known, Kodaira and Chow [4] have shown that every compact, 2-dimensional complex manifold with two independent meromorphic functions is projective algebraic.

1 Complex spaces, pseudoconvexity

1.1 —

Complex spaces are defined as in [10]. We always assume that they are reduced: their local rings contain no nilpotent elements. If X is a complex space, $U = U(x)$ a neighbourhood, $A \subset G \subset \mathbb{C}^n$ an analytic set in a domain G of the space $\mathbb{C}^n$ of the n -dimensional complex numbers, and τ a biholomorphic map $U \rightarrow A$, then (U, τ, A) is called a chart in X , and τ a biholomorphic embedding of U in G .

We always denote by $\mathcal{O} = \mathcal{O}(X)$ the sheaf of germs of locally holomorphic functions on X . If $A \subset X$ is an analytic subset, then we denote by $\mathfrak{m} = \mathfrak{m}(A) \subset \mathcal{O}$ the sheaf of germs of locally holomorphic functions that vanish on A . By a theorem of Cartan, $\mathfrak{m}$ is coherent.

¹Some of the results of the present work were discovered in 1959, and published in [7]. There are, however, some errors in [7]: in Theorem 1, it should, of course, read “[...] such that G is strongly pseudoconvex and A is the maximal compact analytic subset of G ”; furthermore, the criterion in Theorem 2 is only sufficient (see §3.8); Theorem 3 is only proven in the present work in the case where the normal bundle $N(A)$ is weakly negative. — The author has already presented, several times, previously, the example of the complex space X , and, since then, Hironaka has found more interesting examples of complex spaces of this type.

For every subsheaf $\mathcal{J} \subset \mathcal{O}$, let $\mathcal{J}^k$ be the sheaf consisting of germs $f_x = f_{1x} \cdots f_{kx}$, where $f_{1x}, \dots, f_{kx} \in \mathcal{J}_x$ for $x \in X$, $k = 1, 2, \dots$

Now² let $x \in X$, $\mathfrak{m} = \mathfrak{m}(x)$, and $d(x) = \dim_{\mathbb{C}} \mathfrak{m}_x / \mathfrak{m}_x^2$. If $\psi: X \rightarrow \mathbb{C}^n$ is a holomorphic map, then ψ defines, at each point $x \in X$, a homomorphism $\psi_x^*: \mathcal{O}_x(\mathbb{C}^n) \rightarrow \mathcal{O}_x(X)$. This homomorphism maps the maximal ideal $\mathfrak{m}_z \subset \mathcal{O}_z(\mathbb{C}^n)$ to the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_x(X)$. If the induced map $\mathfrak{m}_z / \mathfrak{m}_z^2 \rightarrow \mathfrak{m}_x / \mathfrak{m}_x^2$ is surjective, then we say that ψ is a *regular map* at x . In the case where X is a complex manifold, we see that $\mathfrak{m}_x / \mathfrak{m}_x^2$ is exactly the covariant tangent space of X . Then ψ is regular at $x \in X$ if and only if the Jacobian matrix of ψ at x has rank equal to $\dim_x X$.

We say that a map $\psi: X \rightarrow \mathbb{C}^n$ is *biholomorphic* if it is a bijection that is regular at every point $x \in X$.

(1). *Let x be a point of a complex space X . Then there exists a neighbourhood $U = U(x)$ and a chart (U, τ, A) with $A \subset G$ and $\dim G = d(x)$. If (U, τ, A) is any such chart, and $\psi: U \rightarrow \mathbb{C}^n$ is a regular holomorphic map, then there exists an open neighbourhood $V = V(z)$ of $z = \tau(x)$ in G , and a biholomorphic map $\hat{\psi}: V \rightarrow \mathbb{C}^n$ such that $\psi|_W = \hat{\psi} \circ \tau$ (where $W = \tau^{-1}(V)$).³*

Of course, $\psi|_W$ is then also biholomorphic.

Proof. To prove (1), we may assume that X is an analytic set in a domain $D \subset \mathbb{C}^m$. Let $\hat{\mathfrak{m}}_x$ be the maximal ideal in $\mathcal{O}_x(\mathbb{C}^m)$, and $\mathfrak{i}_x \subset \mathcal{O}_x(\mathbb{C}^m)$ the ideal of germs of holomorphic functions that vanish on $X \subset D$. Let r be the dimension of the image $\mathcal{F}$ of $\mathfrak{i}_x$ under the natural homomorphism $\lambda: \mathfrak{i}_x \rightarrow \hat{\mathfrak{m}}_x / \hat{\mathfrak{m}}_x^2$. Clearly $m = r + d(x)$. Let $f_1, \dots, f_r$ be functions that are holomorphic on a neighbourhood of x , with $f_{vx} \in \mathfrak{i}_x$, so that the elements $\lambda(f_{vx})$ for $v = 1, \dots, r$ span the complex vector space $\mathcal{F}$. Then the rank of the Jacobian matrix of $(f_1, \dots, f_r)$ in X is equal to r . Then, in a neighbourhood $W = W(x)$, the following properties apply:

1. The functions $f_1, \dots, f_r$ are holomorphic on W , and vanish on $X \cap W$;
2. $\hat{G} = \{z \in W \mid f_v(z) = 0 \text{ for } v = 1, 2, \dots, r\}$ is a $d(x)$ -dimensional analytic subset of W that contains no singularities, and which is mapped to a domain in $\mathbb{C}^{d(x)}$ under some biholomorphic map τ .

Now let $A = \tau(X \cap W)$ and $U' = W \cap X$, and we obtain a chart satisfying the required properties.

To prove the second claim of (1), let (U, τ, A) be a chart with $A \subset G$ and $\dim G = d(x)$. We may assume that $U = A$ and that τ is the identity. Then $\lambda(\mathfrak{i}_x) = 0$, since $r = 0$. If $f_1, \dots, f_n$ are holomorphic functions on U that define a holomorphic map $\psi: U \rightarrow \mathbb{C}^n$, and if $\hat{f}_1, \dots, \hat{f}_n$ are holomorphic continuations in an open neighbourhood of x in G , then the rank of the Jacobian matrix of $(\hat{f}_1, \dots, \hat{f}_n)$ at x is equal to $d(x) = \dim G$. There thus exists a neighbourhood $W = W(x)$ in which the $\hat{f}_1, \dots, \hat{f}_n$ are holomorphic and give a biholomorphic map $\psi: W \rightarrow \mathbb{C}^n$. $\square$

By the definition of a complex space, for every point $x \in X$ there is a non-empty system of charts (U, τ, A) such that $x \in U$. As we will show in this section, it is thus possible

²A subscript x always denotes the stalk of the sheaf at the point x . If s is a section, then s_x denotes its value at x . Holomorphic functions and sections in $\mathcal{O}$ are always considered to be the same thing. — If F is a complex-analytic vector bundle, then $\underline{F}$ always denotes the sheaf of germs of locally holomorphic sections in F .

³This statement and its proof were communicated to me by A. Andreotti.

to transfer the concept of strictly plurisubharmonic functions to the setting of complex spaces.

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