# Relative duality in complex analytic geometry

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## Introduction

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Given a $\sigma$-compact analytic space $X$, of bounded dimension, we know how to construct **[10]** a bounded complex $\mathbf{K}_X^\bullet$ of $\mathscr{O}_X$-modules with coherent cohomology, and a (continuous linear) "trace" $\mathbf{T}_X\colon \mathrm{H}_c^0(X;\mathbf{K}_X^\bullet)\to\mathbb{C}$ that can happily provide a duality between the **TODO: separations?** of the spaces $\mathrm{H}_c^p(X;\mathscr{F})$ and $\mathrm{Ext}^{-p}(X;\mathscr{F},\mathbf{K}_X^\bullet)$, for every integer $p$ and every coherent $\mathscr{O}_X$-module $\mathscr{F}$, as long as we endow these space with suitable topologies; similarly, the trace induces a perfect pairing between $\mathrm{H}^p(X;\mathscr{F})$ and $\mathrm{Ext}_c^{-p}(X;\mathscr{F},\mathbf{K}_X^\bullet)$, suitably topologised. The purpose of this article is the study of the compatibility of this duality with direct images: if $f\colon X\to Y$ is a morphism of analytic spaces, then we have to construct a "relative trace" $\mathbf{T}_{X/Y}\in\mathrm{Ext}^0(Y;\mathrm{R}f_!\mathbf{K}_X^\bullet,\mathbf{K}_Y^\bullet)$ (where $f_!$ denotes the direct image with proper support). We then have, for every complex $\mathscr{F}^\bullet$ of $\mathscr{O}_X$-modules with coherent cohomology ($\mathscr{F}^\bullet\in\mathrm{D}_{\mathrm{coh}}(X)$) and for every complex $\mathscr{G}^\bullet$ with bounded-below coherent cohomology ($\mathscr{G}^\bullet\in\mathrm{D}^+_{\mathrm{coh}}(Y)$), a canonical morphism

$$\begin{aligned}\mathrm{R}f_!\mathscr{F}\mathrm{Hom}\big(X;\mathscr{F}^\bullet,\mathrm{R}\mathrm{Hom}(\mathrm{L}f^*\mathrm{R}\mathrm{Hom}(Y;\mathscr{G}^\bullet,\mathbf{K}_Y^\bullet),\mathbf{K}_X^\bullet)\big)&\\ \to\mathrm{R}\mathrm{Hom}(Y;\mathrm{R}f_*\mathscr{F}^\bullet,\mathscr{G}^\bullet)&\end{aligned}\tag{1}$$

that is, in certain cases, an isomorphism (we obtain another formula by swapping $\mathrm{R}f_!$ and $\mathrm{R}f_*$, and more striking formulas by setting $\mathscr{G}^\bullet=\mathbf{K}_Y^\bullet$).

In general, direct images are not coherent, and