$\mathbf{T}$-categories
(Categories in a triple)

Albert Burroni

Note from the translator. *This document is a translation from French of the article*

BURRONI, ALBERT. "T-catégories (catégories dans un triple)". *Cahiers de Topologie et Géométrie différentielle catègoriques* **12** (1971), 215–321. http://www.numdam.org/item?id=CTGDC_1971__12_3_215_0

produced with kind permission from ***TO-DO*** *permissions, and made possible by the efforts of Nathanael Arkor.*

— Tim Hosgood (translator)

Introduction

In friendly homage to José Luis Viviente.

If $\mathbf{T}$ is a triple on a category $\mathscr{E}$, then $\mathbf{T}$-categories are more general structures than $\mathbf{T}$-algebras: they correspond to the passage from "everywhere-defined laws" to "partially defined laws" (in a very broad sense, in fact) and thus also encompass structures as diverse as topologies and categories. *p. 215*

Manes showed, in [Ma], that the category of compact topological spaces is a category of $\mathbf{T}$-algebras, where $\mathbf{T}$ is the triple of ultrafilters on $\mathscr{E} = \mathsf{Set}$. This led Barr, in [Ba], to define "relational $\mathbf{T}$-algebras" so that the category of topological spaces is a category of "relational $\mathbf{T}$-algebras". But Barr seemed disappointed in the fact that, apart from this particular case, and that of preorders relative to the identity triple, there were few examples. Independently, the new system of axioms for topologies that we had given in [Bu] led us to a similar idea (unfortunately with the triple of filters... but our goal was to situate topologies amongst quasi-topologies) that consisted in seeing topologies as a notion analogous to that of preorders (it is indicative, indeed, that the structure of a finite topology is equivalent to that of a finite preorder). But this idea necessitated us to try passing from preorders to categories, and we found, in the definitions of Bénabou [Be] of categories in terms of spans, what we needed in order to define $\mathbf{T}$-categories. We show (**??**proposition:I.2.4]Proposition I.2.4) that the "relational $\mathbf{T}$-algebras" are obtained as a particular case: that of $\mathbf{T}$-preorders (which are to $\mathbf{T}$-categories what preorders are to categories). *p. 216*

The general outline of this article is as follows. Chapter 0 is dedicated to some details of terminology. Chapter I first gives a definition of $\mathbf{T}$-categories that demonstrates their relation to categories: if $\mathscr{E} = \mathsf{Set}$, then the morphisms of a $\mathbf{T}$-category appear, not as arrows going from one object to another object, but as arrows going from one structure on the objects to an object. In this way, a "convergence" in a topology can be represented as an arrow $F \to x$ going from a filter to a point. It is then easy to observe that these structures must be given by the endofunctor of a triple in order to be able to define the properties of reflexivity, transitivite, identity, and associativity of a **T**-category. Then we will expand on some general properties: those that can we obtain without placing any hypotheses on the triple **T**, essentially the existence of projective limits and the fibration of the forgetful functor to $\mathcal{E}$ (which will allow us, in passing, to resolve the problem that consists of universally embedding the forgetful functor of **T**-algebras into a fibrant functor). Other properties, such as the existence of inductive limits, the adjunction with the forgetful functor to **T**-graphs (the analogue of the Stone–Čech–Barr theorem), or the cofibration of the forgetful functor to $\mathcal{E}$ cannot be obtained unless we suppose the triple **T** to be "bounded".

Chapter II is essentially dedicated to two interpretations of **T**-categories, on one hand as monads (or monoids, if one prefers) in the "Kleisli pseudo-category", and on the other as pseudo-algebras in the "bicatgeory of spans".

Chapter III gives various examples, of which the most detailed is that of multicategories (which generalise those defined by Lambek in [La]). For this, we have been led to study in Sections III.1 and III.2 the particular case where **T** is a "strongly cartesian" triple, which not only allows for the construction of free **T**-categories over the usual model of free monoids (see the Appendix), but also gives a manageable description of this structure. We will thus quantitatively improve the examples given by Barr, and we will also later study other useful examples, but this problem appeared less important to us long as our definition seems more natural and has a more general reach. *p. 217*

Chapter IV is dedicated first of all to the description of various particular cases of **T**-functors, for example étale (or "discrete fibration") **T**-functors, and then a generalisation, of **T**-profunctors, that give an approach to **T**-natural transformations.

In a subsequent work, we will define tensor **T**-categories, that will give various "coherence" formulas, in particular those of the "pseudo-categories" introduced in Chapter II. We equally hope to undertake a homological study of these structure.

I would like, in closing, to express my thanks to Madame Bastiani, who encouraged me to work on this subject, and who reread the manuscript, which enabled me to correct numerous imperfections and sometimes errors.

Contents

0 Terminology

Categories generalise, at the same time, monoids and preorders; we thus obtain two types of definitions of categories (and two choices of forgetful functors to sets). The choice between these definitions depends most of all on practical necessities or the generalisations that we wish to obtain. The relationship between these two definitions can be clarified by the notions of monad and polyad of Bénabou [Be]. *p. 219*

We will find ourselves in such a situation due to the various structures that we introduce, and we will call these two types of definitions "global" and "local", respectively. The only goal of the reminders below is the allow us to make precise the terminology and to give us a model of "local" definitions, but, of course, we suppose that the reader knows the elements of the theory of categories and of triples, and this chapter can be quickly skimmed over.

0.1 Categories

A *category* is a quadruple $\mathcal{E} = (|\mathcal{E}|, \mathrm{Hom}_{\mathcal{E}}, i, k)$ satisfying conditions (1) to (6) below:

(1) $|\mathcal{E}|$ is a set; its elements are called *objects of* $\mathcal{E}$.

(2) $\mathrm{Hom}_{\mathcal{E}}$ is a family of sets (that we suppose to be disjoint from one another, and disjoint from $|\mathcal{E}|$, to simplify constructions; but this is rarely the case in practice and, in theory, it is not indispensable). This family is indexed by $|\mathcal{E}|^2$, and the relations

$$e \in |\mathcal{E}|, e' \in |\mathcal{E}|, f \in \mathrm{Hom}_{\mathcal{E}}(e', e)$$

are expressed by simply saying that $f\colon e \to e'$ is a *morphism* (in $|\mathcal{E}|$); we call e the *source of* f and e' the *target of* f.

(3) i is a family of maps of the form

$$i(e)\colon \{\varnothing\} \to \mathrm{Hom}_{\mathcal{E}}(e, e),$$

where the index e runs over the set $|\mathcal{E}|$. We denote by $\mathrm{id}(e)$, id_e, or even simply e, the morphism $i(e)(\varnothing)\colon e \to e$. (This is a sophisticated way of giving a family of morphisms $\mathrm{id}_e\colon e \to e$, where $e \in |\mathcal{E}|$). *p. 220*

(4) k is a family of maps of the form

$$k(e'', e', e)\colon \mathrm{Hom}_{\mathcal{E}}(e'', e') \times \mathrm{Hom}_{\mathcal{E}}(e', e) \to \mathrm{Hom}_{\mathcal{E}}(e'', e),$$

where the index (e'', e', e) runs over the set $|\mathcal{E}|^3$. We generally denote by $g \cdot f$ the composite, i.e. the image under such a map of a pair (g, f).

(5) If $f\colon e \to e'$ is a morphism in $\mathcal{E}$, then

$$f \cdot \mathrm{id}_e = f = \mathrm{id}_{e'} \cdot f.$$

(6) If $f\colon e \to e'$, $g\colon e' \to e''$, and $h\colon e'' \to e'''$ are "consecutive" morphisms in $\mathscr{E}$, then

$$h \cdot (g \cdot f) = (h \cdot g) \cdot f,$$

which allows to us to denote this morphism by $h \cdot g \cdot f$.

A pair $(|\mathscr{E}|, \mathrm{Hom}_{\mathscr{E}})$ satisfying conditions 1 and 2 is called a *graph*; a triple $(|\mathscr{E}|, \mathrm{Hom}_{\mathscr{E}}, i)$ satisfying conditions 1, 2, and 3 is called a *pointed graph*; and in both cases we use the corresponding terminology of conditions 1, 2, and 3.

We say that $F\colon \mathscr{E} \to \mathscr{E}'$ is a *functor* if $\mathscr{E}$ and $\mathscr{E}'$ are categories and F is a pair $(|F|, F_1)$ such that

(1') $|F|\colon |\mathscr{E}| \to |\mathscr{E}'|$ is a map. We denote simply by $F(e)$ the object $|F|(e)$, for all $e \in |\mathscr{E}|$.

(2') F_1 is a family of maps of the form

$$F_1(e', e)\colon\ \mathrm{Hom}_{\mathscr{E}}(e', e) \to \mathrm{Hom}_{\mathscr{E}'}(F(e'), F(e)),$$

where the index (e', e) runs over the set $|\mathscr{E}|^2$. If $f\colon e \to e'$ is a morphism in $\mathscr{E}$, we denote simply by $F(f)$ the morphism $F_1(e', e)(f)$.

(3') $F(\mathrm{id}_e) = \mathrm{id}_{F(e)}$ for all $e \in |\mathscr{E}|$.

(4') $F(g \cdot f) = F(g) \cdot F(f)$, if $f\colon e \to e'$ and $g\colon e' \to e''$ are consecutive morphisms in $\mathscr{E}$.

A *natural transformation* $I\colon F \to F'$, where $F, F'\colon \mathscr{E} \to \mathscr{E}'$ are functors with the same source and target, is given by a family of morphisms in $\mathscr{E}'$ of the form $I(e)\colon F(e) \to F'(e)$, where the index e runs over the set $|\mathscr{E}|$, such that for every morphism $f\colon e \to e'$ in $\mathscr{E}$ we have the relation $I(e') \cdot F(f) = F'(f) \cdot I(e)$. We denote by $\mathscr{E}'^{\mathscr{E}}$ the usual category whose objects are the functors $F\colon \mathscr{E} \to \mathscr{E}'$ and whose morphisms are the natural transformations between these functors. The composition law in $\mathscr{E}'^{\mathscr{E}}$ will generally borrow the notation from that of $\mathscr{E}'$, which risks producing confusion. Indeed, if $F\colon \mathscr{E} \to \mathscr{E}'$ and $G\colon \mathscr{E}' \to \mathscr{E}''$ are consecutive functors, then we denote by $G \cdot F\colon \mathscr{E} \to \mathscr{E}''$ the usual composite. Such a composition extends to natural transformations: if $I\colon F \to F'$ is a morphism in $\mathscr{E}'^{\mathscr{E}}$ and $J\colon G \to G'$ a morphism in $\mathscr{E}''^{\mathscr{E}'}$, then we define a composite $J \cdot I\colon G \cdot F \to G' \cdot F'$ by the formula p. 221

$$(J \cdot I)(e) = J(F'(e)) \cdot G(I(e)) = G'(I(e)) \cdot J(F(e)).$$

But the notation of a dot for this composition risks being confused with the composition in $\mathscr{E}^{\mathscr{E}}$, when, for example, $\mathscr{E} = \mathscr{E}' = \mathscr{E}''$ and the composition in $\mathscr{E}$ is denoted by a dot. Also, even though we generally denote by a point the compositions of "morphisms between structures", we often denote by JI the composite $J \cdot I$ above, so that, for example, we can write

$$JI = JF' \cdot GI = G'I \cdot JF.$$

Similarly, we often write Fe, Ff, and Ie instead of $F(e)$, $F(f)$, and $I(e)$, respectively; this convention is used most of all when the previous convention on the notation JI instead of $J \cdot I$ is in force.

As for *limits*, we adopt the following notation: if $e \in |\mathscr{E}|$, and if $\phi\colon \mathscr{A} \to \mathscr{E}$ is a functor, then we say that $C\colon e \to \phi$ is a *projective cone* if C is a family of morphisms in $\mathscr{E}$, called the *canonical projections of* C, of the form $C(n)\colon e \to \phi(n)$ for all $n \in |\mathscr{A}|$ such that, for every morphism $x\colon n \to n'$ in $\mathscr{A}$, we have that $\phi(x) \cdot C(n) = C(n')$. We say that the projective cone C is a *projective limit*, or simply that e is the *projective limit of* ϕ, if, for every projective cone of the form $C'\colon e' \to \phi$, there exists exactly one morphism $f\colon e' \to e$ in $\mathscr{E}$ such that $C'(n) = C(n) \cdot f$ for all $n \in |\mathscr{A}|$. We call f the *crochet***TO-DO** translator footnote here about "crochet" of C' with respect to the limit C. This terminology is often modified for particular limits: products, fibre products, various diagrams. For example, we say that the following commutative diagram p. 222

$$\begin{tikzcd}
e'' \arrow[d, "g"'] & \pi \arrow[l, "v_2"'] \arrow[d, "v_1"] \\
e & e' \arrow[l, "f"]
\end{tikzcd}$$

is a *cartesian square*, or a *fibre product*[1], or we say that (v_1, v_2), or simply π, is a *fibre product* of (f, g), if the relation $f \cdot f' = g \cdot g'$ in $\mathscr{E}$ implies the existence and universality of a crochet h such that $f' = v_1 \cdot h$ and $g' = v_2 \cdot h$. We use analogous conventions for inductive limits, but the expression "canonical projection" should be replaced by *canonical injection*. We say that $\mathscr{E}$ *admits projective* $\mathscr{A}$*-limits*, for example, if, for every $\phi \in |\mathscr{E}^{\mathscr{A}}|$, there exists a projective limit $C\colon e \to \phi$; and if we have chosen such a limit for all ϕ, we say that $\mathscr{E}$ is endowed with a *canonical* (or *official*) choice of projective $\mathscr{A}$-limits. We use analogous conventions for fibre products, inductive limits, etc. .... Often such a choice is classical and is made tacitly.

We will not cover the notion of a functor *commuting with limits*, nor that of limits commuting with one another; these notions are clear enough.

Recall that, if $C\colon \phi \to e$ is an inductive cone, where $\phi\colon \mathscr{A} \to \mathscr{E}$ is a functor, if $f\colon e' \to e$ is a morphism in $\mathscr{E}$, and if we can form, for all $n \in |\mathscr{A}|$, a fibre product

$$\begin{tikzcd}
\phi(n) \arrow[d, "C(n)"'] & \phi'(n) \arrow[l, "t(n)"'] \arrow[d, "C'(n)"] \\
e & e' \arrow[l, "f"]
\end{tikzcd}$$

then the family $\phi'(n)$ extends to a functor $\phi'\colon \mathscr{A} \to \mathscr{E}$ in a way entirely determined by the condition that $t\colon \phi' \to \phi$ be a natural transformation. We say that the cone $C'\colon \phi' \to e'$ is obtained by the *change of base* f, and we sometimes denote it by f^*C. We say that, in $\mathscr{E}$, the *inductive* $\mathscr{A}$*-limits are universal* if, whenever C is an inductive $\mathscr{A}$-limit and C' exists, then C' is an inductive $\mathscr{A}$-limit.

0.2 Triples

Let $\mathscr{E}$ be a category. We say that $\mathbf{T} = (T, I, K)$ is a *triple* on $\mathscr{E}$ if $T\colon \mathscr{E} \to \mathscr{E}$ is a functor, and $I\colon \mathrm{id}_{\mathscr{E}} \to T$ and $K\colon TT \to T$ are natural transformations such that *p. 223*

$$\begin{aligned} K \cdot TI = T = K \cdot IT \\ K \cdot TK = K \cdot KT \end{aligned} \tag{1}$$

We say that $(M, m)\colon \mathbf{T} \to \mathbf{T}'$ is a *morphism of triples* if $\mathbf{T} = (T, I, K)$ is a triple on $\mathscr{E}$, $\mathbf{T}' = (T', I', K')$ is a triple on another category $\mathscr{E}'$, $M\colon \mathscr{E} \to \mathscr{E}'$ is a functor, and $m\colon T'M \to MT$ is a natural transformation such that

$$\begin{aligned} m \cdot I'M &= MI \\ m \cdot K'M &= MK \cdot mT \cdot T'm. \end{aligned} \tag{2}$$

In general, when there is no risk of confusion, we use the same notation I and K for all triples.

We say that the functor $F\colon \mathscr{E} \to \overline{\mathscr{E}}$ is an *adjoint functor* of the functor $U\colon \overline{\mathscr{E}} \to \mathscr{E}$ if there exist natural transformations $I\colon \mathrm{id}_{\mathscr{E}} \to UF$ and $\overline{I}\colon FU \to \mathrm{id}_{\overline{\mathscr{E}}}$ such that

$$\begin{aligned} U\overline{I} \cdot IU = U \\ \overline{I}F \cdot FI = F. \end{aligned} \tag{3}$$

For all $e \in |\mathscr{E}|$, we call Ie the *adjunction morphism* associated to e; for all $\overline{e} \in |\overline{\mathscr{E}}|$, we call $\overline{I}\overline{e}$ the *coadjunction morphism* associated to $\overline{e}$. We will show that $\mathbf{T} = (UF, I, U\overline{I}F)$ is a triple on $\mathscr{E}$: we say that it is *induced from the adjoint pair* (F, U); it depends on the choice (tacit, in general) of I and $\overline{I}$.

Recall that, if $\mathbf{T}$ is a triple on $\mathscr{E}$, then we denote by $\mathsf{Alg}(\mathbf{T})$ the category of $\mathbf{T}$-*algebras*: a $\mathbf{T}$-algebra on $e \in |\mathscr{E}|$ is a morphism in $\mathscr{E}$ of the form $b\colon Te \to e$ such that

$$\begin{aligned} b \cdot Ie &= e \\ b \cdot Ke &= b \cdot Tb \end{aligned} \tag{4}$$

and a morphism $f\colon b \to b'$ in $\mathsf{Alg}(\mathbf{T})$ is defined by a morphism $f\colon e \to e'$ in $\mathscr{E}$ such that b is an algebra on e, b' is an algebra on e', and

$$f \cdot b = b' \cdot Tf. \tag{5}$$

We denote by $\mathsf{Kl}(\mathbf{T})$ the *Kleisli category* of $\mathbf{T}$: it has the same objects as $\mathscr{E}$, and a morphism $g\colon e \to e'$ in $\mathsf{Kl}(\mathbf{T})$ is a morphism $g\colon Te \to Te'$ in $\mathscr{E}$ such that $g\colon Ke \to Ke'$ is a morphism in $\mathsf{Alg}(\mathbf{T})$. Recall that such a morphism is equivalent to giving a morphism *p. 224* $h\colon e \to Te'$, and that the composite of h with another morphism $h'\colon e' \to Te''$ in $\mathsf{Kl}(\mathbf{T})$ is equal to $Ke'' \cdot Th' \cdot h\colon e \to Te''$. The forgetful functors $\mathsf{Alg}(\mathbf{T}) \to \mathscr{E}$ and $\mathsf{Kl}(\mathbf{T}) \to \mathscr{E}$ (with the latter sending the morphism $g\colon e \to e'$ in $\mathsf{Kl}(\mathbf{T})$ to the morphism $g\colon Te \to Te'$ in $\mathscr{E}$) admit adjoints, and $\mathbf{T}$ is the triple induced form each of these adjoint pairs.

[1] This terminology supposes that we consider the diagram as oriented within the plane.

Recall, finally, that if $\mathbf{T}$ is the triple induced by an adjoint pair (F, U), then we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Kl}(\mathbf{T}) \arrow[r] \arrow[dr] & \overline{\mathcal{E}} \arrow[r] \arrow[d] & \mathsf{Alg}(\mathbf{T}) \arrow[dl] \\
& \mathcal{E} &
\end{tikzcd}$$

with the horizontal arrows being furthermore unique whenever they make the diagram obtained by replacing the three functors to $\mathcal{E}$ with their adjoints commute. The functor $\overline{\mathcal{E}} \to \mathsf{Alg}(\mathbf{T})$ is called the *Eilenberg–Moore functor* associated to (F, U), or simply to U. When this functor is an isomorphism, we say that U is *tripleable*. For example, the Eilenberg–Moore functor $\mathsf{Kl}(\mathbf{T}) \to \mathsf{Alg}(\mathbf{T})$ is fully faithful, but not always injective.

I General properties of T-categories

p. 225

We are going to replace the morphism $b\colon Te \to e$ of a $\mathbf{T}$-algebra, where $\mathbf{T}$ is a triple on a category $\mathcal{E}$, by a span, i.e. by a pair of morphism in $\mathcal{E}$ with the same target

$$b\colon \pi \to e \quad \text{and} \quad a\colon \pi \to Te$$

and define the notion of a $\mathbf{T}$-category by adding data and conditions to this span, in a way that generalises the notion of a "relational $\mathbf{T}$-algebra" of Barr [Ba]. In Chapter II, we will see that this generalisation is natural; below, we will draw inspiration from the "global" definition of a category in order to choose our axioms.

I.1 T-categories

In the rest of this chapter, $\mathcal{E}$ will be a category admitting finite fibre products, and $\mathbf{T} = (T, I, K)$ a triple on $\mathcal{E}$.

A $\mathbf{T}$-*graph* on an object $e \in |\mathcal{E}|$ is a pair (b, a), where

$$b\colon \pi \to e \quad \text{and} \quad a\colon \pi \to Te$$

are morphisms in $\mathcal{E}$.

A *pointed* $\mathbf{T}$-*graph* is a triple (b, a, i) such that (b, a) is a $\mathbf{T}$-graph on e, and $i\colon e \to \pi$ is a morphism in $\mathcal{E}$ such that

$$b \cdot i = \mathrm{id}_e \quad \text{and} \quad a \cdot i = Ie. \tag{1}$$

We then form a fibre product of (a, Tb):

$$\begin{tikzcd}
T\pi \arrow[d, "Tb"'] & \pi_2 \arrow[l, "v_2"'] \arrow[d, "v_1"] \\
Te & \pi \arrow[l, "a"]
\end{tikzcd}$$

The relations

$$a \cdot \mathrm{id}_\pi = a = \mathrm{id}_{Te} \cdot a = Tb \cdot (Ti \cdot a)$$
$$a \cdot (i \cdot b) = Ie \cdot b = Tb \cdot I\pi$$

imply the existence of crochets $i_1, i_2 \colon \pi \to \pi_2$ such that

$$v_1 \cdot i_1 = \mathrm{id}_p \qquad v_2 \cdot i_1 = Ti \cdot a \tag{2}$$

$$v_1 \cdot i_2 = i \cdot b \qquad v_2 \cdot i_2 = I\pi. \tag{3}$$

We then form a fibre product (w_1, w_2) of (v_2, Tv_1). If $k \colon \pi_2 \to \pi$ is a morphism that satisfies p. 226

$$\begin{aligned} b \cdot k &= b \cdot v_1 \\ a \cdot k &= Ke \cdot Ta \cdot v_2 \end{aligned} \tag{4}$$

then the relations

$$a \cdot (k \cdot w_1) = Ke \cdot Ta \cdot v_2 \cdot w_1 = Ke \cdot TTb \cdot Tv_2 \cdot w_2 = Tb \cdot (K\pi \cdot Tv_w \cdot w_2)$$
$$a \cdot (v_1 \cdot w_1) = Tb \cdot Tv_1 \cdot w_2 = Tb \cdot (Tk \cdot w_2)$$

imply the existence of crochets $k_1, k_2 \colon \pi_3 \to \pi_2$ such that

$$v_1 \cdot k_1 = k \cdot w_1 \qquad v_2 \cdot k_1 = K\pi \cdot Tv_2 \cdot w_2 \tag{5}$$

$$v_1 \cdot k_2 = v_1 \cdot w_1 \qquad v_2 \cdot k_2 = Tk \cdot w_2 \tag{6}$$

A **T**-*category*, then, is a quadruple $\theta = (b, a, i, k)$ such that (b, a, i) is a pointed **T**-graph and further satisfying axioms (4), (5), (6), as well as (7) and (8) below:

(7) $k \cdot i_1 = \mathrm{id}_\pi = k \cdot i_2$ ("identity")

(8) $k \cdot k_1 = k \cdot k_2$ ("associativity").

Remark. In reality, θ depends on the choice of the fibre products (v_1, v_2) and (w_1, w_2), p. 227
and we call these pairs the fibre products *associated* to the **T**-category θ; their data is implicit in the quadruple (b, a, i, k). We call b and a the *target morphism* and *source morphism* (respectively) in θ; the object e is denoted $|\theta|$ and called the *object of objects* (or simply the *objects object*), and the object π is called the *object of morphisms* (or simply the *morphisms object*). More generally, π_n for $n = 0, 1, 2, 3$, where $\pi_0 = e$ and $\pi_1 = \pi$, is called the *object of paths of length n* in θ. (All this terminology can also be applied to a **T**-graph.)

When $\mathcal{E}$ is endowed with a canonical choice of fibre products, and (v_1, v_2) and (w_1, w_2) are canonically chosen, we say that θ is a *canonical* (or *official*) **T**-*category*. In the case where $\mathcal{E}$ is endowed with such a choice, it will be implicit that the **T**-category is canonical, and we will explicitly point out if this is not the case. It is evident that to every non-canonical **T**-category there is an associated canonical **T**-category that has the same underlying **T**-graph, and that there is a unique isomorphism (see the definition of morphism that follows) between them. Conversely, if θ is a canonical **T**-category,

$$
\begin{tikzcd}
TTTe \arrow[d, bend right, "KTe"'] \arrow[d, bend left, "TKe"] & TT\pi \arrow[l, "TTa"'] \arrow[dl, "TTb"] \arrow[d, "K\pi"] & T\pi_2 \arrow[l, "Tv_2"'] \arrow[dl, "Tv_1"'] \arrow[dl, bend left, "Tk"] & \pi_3 \arrow[l, "w_2"'] \arrow[dl, "w_1"'] \arrow[dl, bend left, "k_2"] \arrow[dl, bend left=60, "k_1"] \\
TTe \arrow[d, "Ke"] & T\pi \arrow[l, "Ta"'] \arrow[dl, "Tb"'] & \pi_2 \arrow[l, "v_2"'] \arrow[u, "I\pi_2"'] \arrow[dl, "v_1"'] \arrow[dl, bend left, "k"] & \\
Te \arrow[u, bend left, "ITe"] \arrow[u, bend right, "TIe"'] \arrow[ur, bend right, "Ti"'] & \pi \arrow[l, "a"'] \arrow[u, "I\pi"'] \arrow[ur, bend right, "i_1"'] \arrow[ur, bend right=60, "i_2"'] \arrow[dl, "b"'] & & \\
e \arrow[u, "Ie"] \arrow[ur, bend right, "i"'] & & &
\end{tikzcd}
$$

Figure 1.

then every other choice of two fibre products above gives exactly one non-canonical $\mathbf{T}$-category to which it is isomorphic. This remark will be tacitly applied to other analogous structures.

If $\theta = (b, a)$ and $\theta' = (b', a')$ are $\mathbf{T}$-graphs, then a *morphism* f from θ to θ', also denoted $(f_0, f_1)\colon \theta \to \theta'$, is determined by a pair of morphisms $f_0\colon e \to e'$ and $f_1\colon \pi \to \pi'$ in $\mathcal{E}$ between the objects objects and the morphisms objects (respectively) that satisfies the following relations:

$$\begin{aligned} b' \cdot f_1 &= f_0 \cdot b \\ a' \cdot f_1 &= Tf_0 \cdot a. \end{aligned} \tag{1'}$$

We also denote f_0 by $|f|$ and call it the *underlying morphism of* f.

If $\theta = (b, a, i)$ and $\theta' = (b', a', i')$ are pointed $\mathbf{T}$-graphs, then a morphism $(f_0, f_1)\colon \theta \to \theta'$ is defined by a morphism of the underlying $\mathbf{T}$-graphs that further satisfies

$$i' \cdot f_0 = f_1 \cdot i. \tag{2'}$$

p. 228

Finally, let $\theta = (b, a, i, k)$ and $\theta' = (b', a', i', k')$ be $\mathbf{T}$-categories; if (f_0, f_1) defines a morphism of the underlying pointed $\mathbf{T}$-graphs, then there exist crochets $f_2\colon \pi_2 \to \pi_2'$ and $f_3\colon \pi_3 \to \pi_3'$ characterised by the relations

$$\begin{aligned} v_1' \cdot f_2 &= f_1 \cdot v_1 \qquad & v_2' \cdot f_2 &= Tf_1 \cdot v_2 \\ w_1' \cdot f_3 &= f_2 \cdot w_1 \qquad & w_2' \cdot f_3 &= Tf_2 \cdot w_2 \end{aligned}$$

where (v_1, v_2) and (w_1, w_2) are the fibre products associated to θ, and (v_1', v_2') and (w_1', w_2') are the fibre products associated to θ'. Then (f_0, f_1) defines a *morphism of* $\mathbf{T}$-*categories*, or a $\mathbf{T}$-*functor*, $(f_0, f_1)\colon \theta \to \theta'$ if, further,

$$k' \cdot f_2 = f_1 \cdot k. \tag{3'}$$

Let $\mathbf{2}$ be the category consisting of two objects, 0 and 1, and a single morphism $0 \to 1$ (apart from the two identity morphisms). An object of $\mathcal{E}^{\mathbf{2}}$ can be identified with a morphism in $\mathcal{E}$, and $(g, g')\colon f \to f'$ can be associated to a morphism in $\mathcal{E}^{\mathbf{2}}$ if g, g', f, f' are morphisms in $\mathcal{E}$ such that $f' \cdot g' = g \cdot f$. If $\mathcal{E}$ admits fibre products, then so too does $\mathcal{E}^{\mathbf{2}}$, and, if $\mathbf{T} = (T, I, K)$ is a triple on $\mathcal{E}$, then it induces a triple $\mathbf{T}^{\mathbf{2}}$ on $\mathcal{E}^{\mathbf{2}}$, where $\mathbf{T}^{\mathbf{2}} = (T^{\mathbf{2}}, I^{\mathbf{2}}, K^{\mathbf{2}})$, just as for any category of presheaves (i.e. of functors) with values in $\mathcal{E}$. If $f\colon e \to e'$ is a morphism in $\mathcal{E}$, and thus an object in $\mathcal{E}^{\mathbf{2}}$, then $T^{\mathbf{2}} f = Tf$, and $I^{\mathbf{2}} f$ and $K^{\mathbf{2}} f$ are the following morphisms:

$$\begin{aligned} I^{\mathbf{2}} f &= (Ie', Ie)\colon f \to Tf \\ K^{\mathbf{2}} f &= (Ke', Ke)\colon TTf \to Tf. \end{aligned}$$

If $(g, g')\colon f \to f'$ is a morphism in $\mathcal{E}^{\mathbf{2}}$, then its image under $T^{\mathbf{2}}$ is, by definition, the morphism $(Tg, Tg')\colon Tf \to Tf'$ in $\mathcal{E}^{\mathbf{2}}$.

Proposition I.1.1. *A* $\mathbf{T}^{\mathbf{2}}$*-category is identifiable with a morphism of* $\mathbf{T}$*-categories.*

Proof. The proof, which is elementary, is left to the reader; we simply note that, if $(f_0, f_1)\colon \theta \to \theta'$ is a morphism of $\mathbf{T}$-categories, then f_0 is the objects object, and f_1 the morphisms object, of the associated $\mathbf{T}^{\mathbf{2}}$-category. □

We can make analogous remarks for $\mathbf{T}$-graphs and pointed $\mathbf{T}$-graphs. *p. 229*

If $\mathcal{E}$ is endowed with official finite fibre products, then we denote by $\mathsf{Gr}(\mathbf{T})$ and $\mathsf{Cat}(\mathbf{T})$ the categories whose objects are the official $\mathbf{T}$-graphs and the official $\mathbf{T}$-categories (respectively). If $\mathcal{E}$ admits finite fibre products without us having made an official choice, then we sometimes denote by $\mathsf{Gr}(\mathbf{T})$ and $\mathsf{Cat}(\mathbf{T})$ the categories (which are equivalent to the above) whose objects are all $\mathbf{T}$-graphs and all $\mathbf{T}$-categories (respectively). The composition laws in these categories are evident.

I.2 T-preorders

A *regular* $\mathbf{T}$-graph is a $\mathbf{T}$-graph (b, a) such that the relations

$$b \cdot f = b \cdot f' \quad \text{and} \quad a \cdot f = a \cdot f'$$

imply that $f = f'$. A $\mathbf{T}$-*preorder* is a $\mathbf{T}$-category whose underlying graph is regular.

Proposition I.2.2. *If (b, a) is a regular $\mathbf{T}$-graph, e the objects object, π the morphisms object, π_2 the source of a fibre product of (a, Tb), and $i\colon e \to \pi$ and $k\colon \pi_2 \to \pi$ morphisms in $\mathcal{E}$ satisfying only conditions (1) to (4) of §I.1, then (b, a, i, k) is a $\mathbf{T}$-preorder. Furthermore, conditions (2') and (3') are consequences of (1') for a morphism into a $\mathbf{T}$-preorder.*

Proof. We have to show that the axioms of identity and associativity, (7) and (8) of §I.1, are automatically satisfied. Since (b, a) is regular, the relations

$$k \cdot i_1 = \pi = k \cdot i_2$$
$$k \cdot k_1 = k \cdot k_2$$

lead to the following equalities:

(α) $b \cdot k \cdot i_1 = b = b \cdot k \cdot i_2$

(β) $a \cdot k \cdot i_1 = a = a \cdot k \cdot i_2$

(γ) $b \cdot k \cdot k_1 = b \cdot k \cdot k_2$

(δ) $a \cdot k \cdot k_1 = a \cdot k \cdot k_2$.

Indeed, *p. 230*

$$b \cdot k \cdot i_1 \underset{(4)}{=} b \cdot v_1 \cdot i_1 \underset{(2)}{=} b$$

and similarly

$$b \cdot k \cdot i_2 = b \cdot v_1 \cdot i_2 \underset{(3)}{=} b \cdot i \cdot b \underset{(1)}{=} b$$

which proves (α). Next,

$$\begin{aligned} a \cdot k \cdot i_1 &\underset{(4)}{=} Ke \cdot Ta \cdot v_2 \cdot i_1 = Ke \cdot Ta \cdot Ti \cdot a \\ &= Ke \cdot T(a \cdot i) \cdot a \underset{(1)}{=} Ke \cdot TIe \cdot a \\ &= a \end{aligned}$$

and

$$
\begin{aligned}
a \cdot k \cdot i_2 &= Ke \cdot Ta \cdot v_2 \cdot i_2 \\
&= Ke \cdot Ta \cdot I\pi = Ke \cdot ITe \cdot a \\
&= a
\end{aligned}
$$

which proves (β). Then

$$
\begin{aligned}
b \cdot k \cdot k_1 &= b \cdot v_1 \cdot k_1 = b \cdot k \cdot w_1 \\
&= b \cdot v_1 \cdot w_1 = b \cdot v_1 \cdot k_2 \\
&= b \cdot k \cdot k_2
\end{aligned}
$$

whence (γ). Finally,

$$
\begin{aligned}
a \cdot k \cdot k_1 &= Ke \cdot Ta \cdot v_2 \cdot k_1 = Ke \cdot Ta \cdot K\pi \cdot Tv_2 \cdot w_2 \\
&= Ke \cdot KTe \cdot TTa \cdot Tv_2 \cdot w_2 = Ke \cdot TKe \cdot TTa \cdot Tv_2 \cdot w_2 \\
&= Ke \cdot T(Ke \cdot Ta \cdot v_2) \cdot w_2 = Ke \cdot T(a \cdot k) \cdot w_2 \\
&= Ke \cdot Ta \cdot Tk \cdot w_2 = Ke \cdot Ta \cdot v_2 \cdot k_2 \\
&= a \cdot k \cdot k_2
\end{aligned}
$$

which proves (δ). The verification of the last claim of the proposition is easy. □

This proposition allows us to identify a $\mathbf{T}$-preorder with its underlying $\mathbf{T}$-graph, and to sometimes write (b, a) instead of (b, a, i, k) for a $\mathbf{T}$-preorder. Amongst all $\mathbf{T}$-preorders, many interesting cases can be distinguished by properties of by a and b (see the examples in ??). We limit ourselves to noting the following fact:

Proposition I.2.3. *A $\mathbf{T}$-algebra b over e can be identified with a $\mathbf{T}$-preorder (b, a) such that $a = \mathrm{id}_{Te}$, and conversely.*

Proof. This is completely evident, since the relation $a = \mathrm{id}_{Te}$ implies that $i = Ie$, $Ta = v_2 = \mathrm{id}_{TTe}$, and $k = Ke$. □

p. 231

We denote by $\mathsf{Ord}(\mathbf{T})$ and $\mathsf{Alg}(\mathbf{T})$ the full subcategories of the category $\mathsf{Gr}(\mathbf{T})$ whose objects are the $\mathbf{T}$-preorders and the $\mathbf{T}$-algebras (respectively). We have just seen that $\mathsf{Alg}(\mathbf{T})$ can be identified with the usual category of $\mathbf{T}$-algebras. Furthermore, when $\mathcal{E} = \mathsf{Set}$ is the category of sets associated to a universe, we have:

Proposition I.2.4. *If $\mathcal{E} = \mathsf{Set}$, then the category $\mathsf{Ord}(\mathbf{T})$ is equivalent to the category of "relational $\mathbf{T}$-algebras" of Barr [Ba].*

Proof. A "relational $\mathbf{T}$-prealgebra" is a relation $r \colon Te \to e$. It is defined by a subset π of $e \times Te$, which defines a regular $\mathbf{T}$-graph (b, a) by taking b and a to be the composite of the inclusion $\pi \to e \times Te$ with the projections $e \times Te \to e$ and $e \times Te \to Te$ (respectively). This is a "relational $\mathbf{T}$-algebra" if it further satisfies the conditions:

(1) $\mathrm{id}_e \subset r \cdot Ie$

(2) $r \cdot \tilde{T}r \subset r \cdot Ke$

where $\tilde{T}r\colon TTe \to Te$ is the relation defined by the set

$$\hat{\pi} = \{(Tb(z), Ta(z)) \mid z \in T\pi\} \subset Te \times TTe.$$

We identify every map $f\colon e \to e'$ with a relation, and we denote by

$$\tilde{f} = \{(f(x), x) \mid x \in e\} \subset e' \times e$$

its "graph". We will show that (b, a) is a $\mathbf{T}$-preorder.

For each $x \in e$, set $i(x) = (x, Ie(x))$; we have that $i(x) \in \pi$, since, by (1), $(x, x) \in \mathrm{id}_e$ implies that there exists $y \in Te$ such that

$$(x, y) \in r \quad \text{and} \quad (y, x) \in Ie$$

and, since Ie is a map, we have that $y = Ie(x)$; thus $i(x) \in \pi$, and $i\colon e \to \pi$ is a map such that

$$b \cdot i = \mathrm{id}_e \quad \text{and} \quad a \cdot i = Ie.$$

An element $((x, y), m)$ of the fibre product π_2 of (a, Tb) is characterised by the relations $(x, y) \in \pi$, $m \in T\pi$, and $y = Tb(m)$. Set

$$k((x, y), m) = (x, Ke \cdot Ta(m))$$

which is an element of π. Indeed, since $(Tb(m), Ta(m)) \in \hat{\pi}$, condition (2) implies that $(x, Ta(m)) \in r \cdot Ke$, and so there exists $y \in Te$ such that $(x, y) \in \pi$ and $(y, Ta(m)) \in Ke$. Since Ke is an application, we deduce that $y = Ke \cdot Ta(m)$, which shows that $k((x, y), m) \in \pi$, and that the map $k\colon \pi_2 \to \pi$ satisfies the conditions *p. 232*

$$b \cdot k = b \cdot v_1 \quad \text{and} \quad a \cdot k = Ke \cdot Ta \cdot v_2.$$

This, by **which proposition number?** proves that (b, a) is a $\mathbf{T}$-preorder. □

I.3 Properties (projective limits, fibrations, . . .)

We have a sequence of forgetful functors

$$\mathsf{Alg}(\mathbf{T}) \to \mathsf{Ord}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T}) \to \mathcal{E}. \tag{$*$}$$

The first two are full inclusions, the third is faithful and the fourth is (in general) not.

Proposition I.3.5. *If $\mathcal{E}$ admits projective $\mathcal{A}$-limits, then so too do each of the categories appearing in the sequence $(*)$, and the functors in this sequence commute with these limits.*

Proof. We will first show how to construct a projective limit for a functor $\phi\colon \mathcal{A} \to \mathsf{Cat}(\mathbf{T})$. To each *symbol* of an object x in Figure 1 there corresponds an evident functor, which we denote by $U_x\colon \mathsf{Cat}(\mathbf{T}) \to \mathcal{E}$, that sends a $\mathbf{T}$-category θ to the object denoted x in **which figure?**. Thus

$$U_e(\theta) = e$$
$$U_\pi(\theta) = \pi$$
$$U_Te(\theta) = Te$$

etc., and to each *symbol* of a morphism $\xi\colon x \to x'$ in Figure 1 associates a natural transformation $U_\xi\colon U_x \to U_{x'}$ such that $U_\xi(\theta) = \xi$. We set

$$\hat{x} = U_x \cdot \phi \quad \text{and} \quad \hat{\xi} = U_\xi \cdot \phi;$$

then, for all $n \in |\mathscr{A}|$, we have that

$$\phi(n) = \big(\hat{b}(n), \hat{a}(n), \hat{i}(n), \hat{k}(n).\big)$$

If $\hat{x}\colon \mathscr{A} \to \mathscr{E}$ is a functor, then we denote by $p_n^x\colon \lim \hat{x} \to \hat{x}(n)$ the n-th canonical projection of this limit. Similarly, if $\hat{\xi}\colon \hat{x} \to \hat{x}'$ is a natural transformation, then we denote by $\lim \hat{\xi}\colon \lim \hat{x} \to \lim \hat{x}'$ the unique morphism in $\mathscr{E}$ such that $\lim \hat{\xi} \cdot p_n^x = p_n^{x'} \cdot \hat{\xi}(n)$ for all $n \in |\mathscr{A}|$.[2] We set $e = \lim \hat{e}$. *p. 233*

Let m^{Te} be the unique morphism (again called the crochet) that comes from the property of the projective limit $\lim \widehat{Te}$, such that $p_n^{Te} \cdot m^{Te} = Tp_n^e$ for all $n \in |\mathscr{A}|$. We take a fibre product of $(\lim \hat{a}, m^{Te})$:

$$\begin{tikzcd}
\lim \widehat{Te} & Te \arrow[l, "m^{Te}"'] \\
\lim \hat{\pi} \arrow[u, "\lim \hat{a}"] & \pi \arrow[l, "m^{\pi}"] \arrow[u, "a"']
\end{tikzcd}$$

We have that $m^{Te} \cdot Ie = \lim \widehat{Ie}$. The $\mathbf{T}$-graph (b, a), where $b = \lim \hat{b} \cdot m^\pi$ and a is the projection defined above, extends to a $\mathbf{T}$-category. Indeed, the relations

$$\lim \hat{a} \cdot \lim \hat{i} = \lim \widehat{Ie} = m^{Te} \cdot Ie$$

imply the existence of a bracket $i\colon e \to \pi$ characterised by the relations

$$m^\pi \cdot i = \lim \hat{i} \quad \text{and} \quad a \cdot i = Ie$$

and then (b, a, i) is a pointed $\mathbf{T}$-graph.

Let $m^{T\pi}\colon T\pi \to \lim \widehat{T\pi}$ be the crochet characterised by the relations $p_n^{T\pi} \cdot m^{T\pi} = Tp_n^\pi$; we will show that $\lim \widehat{Tb} \cdot m^{T\pi} = m^{Te} \cdot Tb$ by obtaining equalities whenever we compose on the left by the p_n^{Te}. It thus follows, since projective limits of fibre products are again fibre products, that there exists a crochet $m^{\pi_2}\colon \pi_2 \to \lim \hat{\pi}_2$ such that

$$\begin{aligned}
\lim \hat{v}_2 \cdot m^{\pi_2} &= m^{T\pi} \cdot v_2 \\
\lim \hat{v}_1 \cdot m^{\pi_2} &= m^{\pi} \cdot v_1
\end{aligned}$$

where (v_1, v_2) is the fibre product of (a, Tb). The relations

$$\begin{aligned}
\lim \hat{a} \cdot \lim \hat{k} \cdot m^{\pi_2} &= \lim \widehat{Ke} \cdot (\lim \widehat{Ta} \cdot \lim \hat{v}_2 \cdot m_2^\pi) \\
&= \lim \widehat{Ke} \cdot m^{TTe} \cdot Ta \cdot v_2 \\
&= m^{Te} \cdot Ke \cdot Ta \cdot v_2
\end{aligned}$$

[2] In a future work, we will consider Figure 1 as the "sketch" of $\mathbf{T}$-categories, with the latter being exactly the "realisations" of this "sketch" endowed with "typifications" (see the terminology of [Bu], for example).

imply the existence of $k\colon \pi_2 \to \pi$ such that

$$a \cdot k = Ke \cdot Ta \cdot v_2$$
$$m^\pi \cdot k = \lim \hat{k} \cdot m^{\pi_2}.$$

It thus follows that $b \cdot k = b \cdot v_1$. It remains only to show that (b, a, i, k) satisfies the axioms of identity and associativity, which we leave to the reader. *p. 234*

To see that (b, a, i, k) is indeed a projective limit, suppose that (b', a', i', k') is a $\mathbf{T}$-category, (v_1', v_2') and (w_1', w_2') the associated fibre products, and e', π', π_2', π_3' the objects of paths of length $n = 0, 1, 2, 3$ (respectively); consider a projective cone from this structure to the functor $\phi\colon \mathscr{A} \to \mathsf{Cat}(\mathbf{T})$. For each symbol of an object x in Figure 1, we denote by $q_n^x\colon x' \to \hat{x}(n)$ the n-th projection, and by q^x the crochet characterised by $q_n^x = p_n^x \cdot q^x$ for all $n \in |\mathscr{A}|$. By composing with the p_n^{Te} on the left, we note that $m^{Te} \cdot Tq^e = q^{Te}$, and so

$$\lim \hat{a} \cdot q^\pi = q^{Te} \cdot a' = m^{Te} \cdot Tq^e \cdot a'$$

which implies the existence of a crochet $\overline{q}\colon \pi' \to \pi$ characterised by

$$a \cdot \overline{q} = Tq^e \cdot a' \quad \text{and} \quad m^\pi \cdot \overline{q} = q^\pi.$$

We then easily finish the proof by showing that $(q^e, \overline{q})$ defines a homomorphism of $\mathbf{T}$-categories.

The construction of projective limits of $\mathbf{T}$-graphs and of pointed $\mathbf{T}$-graphs is, evidently, "underlying" to that which we have just done. It is then immediate that the forgetful functor $\mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T})$ commutes with these limits. Finally, if in the above construction of $\lim \phi$ the $\mathbf{T}$-graphs $(b(n), a(n))$ were regular then so too would be (b, a), which makes explicit the projective limits in $\mathsf{Ord}(\mathbf{T})$, and if the $a(n)$ were simply identities then so to would be a; thus the "full inclusion" functor $\mathsf{Alg}(\mathbf{T}) \to \mathsf{Ord}(\mathbf{T})$ commutes with these limits. □

Corollary. *If $\mathbf{T}$ commutes with projective $\mathscr{A}$-limits then the above construction simplifies, and $x = \lim \hat{x}$ for every symbol x.*

Proof. This is completely immediate, since the projective $\mathscr{A}$-limits of fibre products are again fibre products (by the commutativity of projective limits with themselves). □

If $U\colon \overline{\mathscr{E}} \to \mathscr{E}$ is a functor, we say that a morphism $\overline{f}\colon \overline{e}' \to \overline{e}$ in $\overline{\mathscr{E}}$ is *U-cartesian* (or is a *U-injection*, in the terminology of [Eh]) if, for every morphism $\overline{g}\colon \overline{e}'' \to \overline{e}$ in $\overline{\mathscr{E}}$ and every relation $U(\overline{g}) = U(\overline{f}) \cdot h$ in $\mathscr{E}$, there exists exactly one morphism $\overline{h}\colon \overline{e}'' \to \overline{e}'$ in $\overline{\mathscr{E}}$ such that we have $h = U(\overline{h})$. We say that U is *fibrant* if, for all $\overline{e} \in |\overline{\mathscr{E}}|$ and every morphism $f\colon e' \to U(\overline{e})$ in $\mathscr{E}$, there exists a U-cartesian morphism $\overline{f}\colon \overline{e}' \to \overline{e}$ such that $f = U(\overline{f})$; we sometimes denote by $f^*\overline{e}$ the object $\overline{e}'$. If $U'\colon \overline{\mathscr{E}}' \to \mathscr{E}$ is another fibrant *p. 235* functor, we say that a functor $V\colon \overline{\mathscr{E}} \to \overline{\mathscr{E}}'$ is *compatible with these fibrations* if it sends every U-cartesian morphism to a U'-cartesian morphism. With the above notation, we have an isomorphism $V(f^*\overline{e}) \to f^*V(\overline{e})$.

Proposition I.3.6. *The functors $\mathscr{X} \to \mathscr{E}$ coming from all the composites of $(*)$ for $\mathscr{X} = \mathsf{Ord}(\mathbf{T})$, $\mathsf{Cat}(\mathbf{T})$, and $\mathsf{Gr}(\mathbf{T})$ are fibrant, and these fibrations are respected by the functors from $(*)$ between these categories.*

Proof. Let $\theta = (b, a)$ be a $\mathbf{T}$-graph, and $f_0 \colon e' \to e$ a morphism in $\mathcal{E}$, where $e = |\theta|$. We take a projective limit of the diagram (the solid arrows)

$$
\begin{tikzcd}
Te & Te' \arrow[l, "Tf_0"'] \\
\pi \arrow[u, "a"] \arrow[d, "b"'] & \pi' \arrow[l, dashed, "f_1"'] \arrow[u, dashed, "a'"'] \arrow[d, dashed, "b'"] \\
e & e' \arrow[l, "f_0"]
\end{tikzcd}
$$

(the existence of such a limit follows from that of finite fibre products). We obtain projections (the dashed arrows) b', f_1, and a'. We thus obtain a $\mathbf{T}$-graph $\theta' = (b', a')$ and a morphism

$$(f_0, f_1) \colon \theta' \to \theta$$

in $\mathsf{Gr}(\mathbf{T})$. Let $(f'_0, f'_1) \colon \theta'' \to \theta$ be another morphism, where $\theta'' = (b'', a'')$, and let $g_0 \colon e'' \to e'$ be such that $e'' = |\theta''|$ and $f'_0 = f_0 \cdot g_0$; we will construct a morphism $(g_0, g_1) \colon \theta'' \to \theta'$ such that $f'_1 = f_1 \cdot g_1$. Consider the commutative diagram

$$
\begin{tikzcd}
Te & Te' \arrow[l, "Tf'_0"'] \\
\pi \arrow[u, "a"] \arrow[d, "b"'] & \pi'' \arrow[l, "f'_1"'] \arrow[u, "Tg_0 \cdot a'"'] \arrow[d, "g_0 \cdot b'"] \\
e & e' \arrow[l, "f'_0"]
\end{tikzcd}
$$

The comparison with the projective limit gives a crochet $g_1 \colon \pi'' \to \pi'$ such that *p. 236*

$$
\begin{aligned}
a' \cdot g_1 &= Tg_0 \cdot a'' \\
f_1 \cdot g_1 &= f'_1 \\
b' \cdot g_1 &= g_0 \cdot b''.
\end{aligned}
$$

It thus follows that $(g_0, g_1) \colon \theta'' \to \theta'$ is a morphism in $\mathsf{Gr}(\mathbf{T})$, and that $(f_0, f_1) \colon \theta' \to \theta$ is a U-cartesian morphism for the forgetful functor $U \colon \mathsf{Gr}(\mathbf{T}) \to \mathcal{E}$. We set $\theta' = f_0^* \theta$. Note also that if θ is regular then so too is θ'.

Now support that $\theta = (b, a)$ underlies a $\mathbf{T}$-category $\overline{\theta} = (b, a, i, k)$; we will construct a $\mathbf{T}$-category $\overline{\theta}' = (b', a', i', k')$ such that $f_0^* \theta = (b', a')$. Consider the commutative diagrams

$$
\begin{tikzcd}
Te & Te' \arrow[l, "Tf_0"'] \\
\pi \arrow[u, "a"] \arrow[d, "b"'] & e' \arrow[l, "i\cdot f_0"'] \arrow[u, "Ie'"'] \arrow[d, "\mathrm{id}_{e'}"] \\
e & e' \arrow[l, "f_0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Te & Te' \arrow[l, "Tf_0"'] \\
\pi \arrow[u, "a"] \arrow[d, "b"'] & \pi'_2 \arrow[l, "i\cdot f_0"'] \arrow[u, "Ke'\cdot Ta'\cdot v'_2"'] \arrow[d, "b'\cdot v'_1"] \\
e & e' \arrow[l, "f_0"]
\end{tikzcd}
$$

where $f_2\colon \pi'_2 \to \pi_2$ is defined above, from (f_0, f_1); these give two crochets

$$i'\colon e' \to \pi' \quad \text{and} \quad k'\colon \pi'_2 \to e'$$

such that (b', a', i', k') is the desired $\mathbf{T}$-category, denoted $\overline{\theta}' = f_0^*\overline{\theta}$. Furthermore, $(f_0, f_1)\colon \overline{\theta}' \to \overline{\theta}$ is a morphism in $\mathsf{Cat}(\mathbf{T})$. We can then show that, if the $\mathbf{T}$-graph θ'' from the previous proof underlies a $\mathbf{T}$-category $\overline{\theta}''$, and if $(f'_0, f'_1)\colon \overline{\theta}'' \to \overline{\theta}$ is a morphism in $\mathsf{Cat}(\mathbf{T})$, then so too is $(g_0, g_1)\colon \overline{\theta}'' \to \overline{\theta}'$. Thus $(f_0, f_1)\colon \overline{\theta}' \to \overline{\theta}$ is cartesian. Finally, if $\overline{\theta}$ is a $\mathbf{T}$-preorder, then so too is $\overline{\theta}'$. $\square$

Now let $\mathbf{T}' = (T', I', K')$ be a triple on another category $\mathscr{E}'$ that admits finite fibre products. Let $(M, m)\colon \mathbf{T} \to \mathbf{T}'$ be a morphism of triples (see §I.3), where $M\colon \mathscr{E} \to \mathscr{E}'$ is a functor that respects finite fibre products. Finally, suppose that $\mathscr{E}$ and $\mathscr{E}'$ admit projective $\mathscr{A}$-limits, and that M respects these projective limits. Set $\mathbf{M} = (M, m)$. *p. 237*

Proposition I.3.7. *We can construct functors* $\mathsf{Alg}(\mathbf{M})$, $\mathsf{Ord}(\mathbf{M})$, *etc. such that the following diagram commutes:*

$$
\begin{tikzcd}
\mathsf{Alg}(\mathbf{T}) \arrow[r] \arrow[d, "\mathsf{Alg}(\mathbf{M})"'] & \mathsf{Ord}(\mathbf{T}) \arrow[r] \arrow[d, "\mathsf{Ord}(\mathbf{M})"'] & \mathsf{Cat}(\mathbf{T}) \arrow[r] \arrow[d, "\mathsf{Cat}(\mathbf{M})"'] & \mathsf{Gr}(\mathbf{T}) \arrow[r] \arrow[d, "\mathsf{Gr}(\mathbf{M})"'] & \mathscr{E} \arrow[d, "\mathbf{M}"] \\
\mathsf{Alg}(\mathbf{T}') \arrow[r] & \mathsf{Ord}(\mathbf{T}') \arrow[r] & \mathsf{Cat}(\mathbf{T}') \arrow[r] & \mathsf{Gr}(\mathbf{T}') \arrow[r] & \mathscr{E}'
\end{tikzcd}
$$

These functors respect projective $\mathscr{A}$-limits and the fibrations defined in Proposition I.3.6.

Proof. Let $\theta = (b, a)$ be a $\mathbf{T}$-graph, and consider the fibre product

$$
\begin{tikzcd}
MTe & T'Me \arrow[l, "me"'] \\
M\pi \arrow[u, "Ma"] & \pi' \arrow[l, "m'"] \arrow[u, "a'"']
\end{tikzcd}
$$

Then $\theta' = (b', a')$, where $b' = Mb \cdot m'$, is a $\mathbf{T}'$-graph. This correspondence extends to a functor $\mathsf{Gr}(\mathbf{M})\colon \mathsf{Gr}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T}')$.

Now, if (b, a, i, k) is a $\mathbf{T}$-category, then we have a crochet $i'\colon Me \to \pi'$ characterised by

$$m' \cdot i' = Mi \quad \text{and} \quad a' \cdot i' = I'Me.$$

Set $m'' = m\pi \cdot T'm'$; we will show that $MTb \cdot m'' = me \cdot T'b'$, so that there exists a morphism $m'''\colon \pi_2' \to M\pi_2$ that makes the following diagram commute:

$$\begin{tikzcd}
MT\pi & T'\pi' \arrow[l, "m''"'] \\
M\pi_2 \arrow[u, "Mv_2"] \arrow[d, "Mv_1"'] & \pi_2' \arrow[l, "m'''"'] \arrow[u, "v_2'"'] \arrow[d, "v_1'"] \\
M\pi & \pi' \arrow[l, "m'"]
\end{tikzcd}$$

(with the data of π_2', π_3', v_1', etc. being defined as per usual from the $\mathbf{T}'$-graph (b', a')). The morphism $k'\colon \pi_2' \to \pi'$ is the crochet characterised by the relations

$$\begin{aligned} m' \cdot k' &= Mk \cdot m''' \\ a' \cdot k' &= K'Me \cdot T'a' \cdot v_2'. \end{aligned}$$

We can easily show that (b', a', i', k') is a $\mathbf{T}'$-category, and that the correspondence thus *p. 238*
defined extends to a functor

$$\mathsf{Cat}(\mathbf{M})\colon \mathsf{Cat}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T}').$$

The rest of the proof is purely technical, although annoying. □

Remark. As an exercise, it is interesting to examine the case where $M = \mathrm{id}_{\mathscr{E}}$, $m = I$, and $\mathbf{T} = (\mathrm{id}_{\mathscr{E}}, \mathrm{id}_{\mathscr{E}}, \mathrm{id}_{\mathscr{E}})$: there is an $\mathscr{E}$-category underlying every $\mathbf{T}$-category (in the terminology of §III.2).

We will later study the question of whether or not all functors in the sequence $(*)$ admit adjoints and coadjoints; we will limit ourselves below to only some of them. Recall that $\mathsf{Alg}(\mathbf{T}) \to \mathscr{E}$ has an adjoint. Suppose that $\mathscr{E}$ admits finite projective limits.

Proposition I.3.8. *The forgetful functors $\mathscr{X} \to \mathscr{E}$, for $\mathscr{X} = \mathsf{Gr}(\mathbf{T})$, $\mathsf{Cat}(\mathbf{T})$, and $\mathsf{Ord}(\mathbf{T})$, admit adjoints and coadjoints which are further sections of these forgetful functors, and commute with the sequence of functors $\mathsf{Ord}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T})$.*

Proof. Let $e \in |\mathscr{E}|$. Set

$$D(e) = (\mathrm{id}_e, Ie) \quad \text{and} \quad G(e) = (p_1, p_2)$$

with the latter being formed of the canonical projections of a product of (e, Te). Then $D(e)$ and $G(e)$ are $\mathbf{T}$-graphs, which determine $\mathbf{T}$-categories

$$\overline{D}(e) = (\mathrm{id}_e, Ie, \mathrm{id}_e, \mathrm{id}_e) \quad \text{and} \quad \overline{G}(e) = (p_1, p_2, i_0, k_0)$$

where i_0 and k_0 are the crochets characterised by the relations

$$\begin{aligned}
p_1 \cdot i_0 &= \mathrm{id}_e \\
p_2 \cdot i_0 &= Ie \\
p_1 \cdot k_0 &= p_1 \cdot v_1 \\
p_2 \cdot k_0 &= Ke \cdot Tp_2 \cdot v_2
\end{aligned}$$

where (v_1, v_2) is a fibre product of (p_2, Tp_1). Furthermore, we note that $\overline{D}(e)$ and $\overline{G}(e)$ are $\mathbf{T}$-preorders. It is easy to show that we can thus define an adjoint D and a coadjoint G to the functor $\mathsf{Gr}(\mathbf{T}) \to \mathcal{E}$, and an adjoint $\overline{D}$ and a coadjoint $\overline{G}$ to the functor $\mathsf{Cat}(\mathbf{T}) \to \mathcal{E}$, which, by restriction, gives an adjoint and a coadjoint to the functor $\mathsf{Ord}(\mathbf{T}) \to \mathcal{E}$. □

p. 239

We call $\overline{D}(e)$ and $\overline{G}(e)$ the *discrete* $\mathbf{T}$*-category* and the *coarse* $\mathbf{T}$*-category* on e (respectively).

Finally, we examine the case of the inclusion $\mathsf{Ord}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T})$.

Suppose that $\mathcal{E}$, as well as satisfying the existence of finite fibre products, also satisfies the following conditions:

(I) Every morphism f in $\mathcal{E}$ decomposes into a monomorphism m and an epimorphism p, as $f = m \cdot p$.

(II) Every epimorphism p of $\mathcal{E}$ is a retraction, i.e. there exists a morphism s in $\mathcal{E}$ (called a *section of* p) such that $p \cdot s = \mathrm{id}_e$, where e is the target of p.

By (II), the decomposition of f in (I) is unique up to isomorphism; we call (m, p) a *canonical decomposition* of f. More generally, if $f = m' \cdot p'$, and if m' is a monomorphism, then there exists exactly one morphism g such that $m' \cdot g = m$ and $g \cdot p = p'$.

Proposition I.3.9. *If $\mathcal{E}$ satisfies the above hypotheses, then the inclusion functor $\mathsf{Ord}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T})$ admits an adjoint that is compatible with the forgetful functors to $\mathcal{E}$.*

Proof. To each $\mathbf{T}$-graph $\theta = (b, a)$ on e, we associate, thanks to the canonical decomposition of the crochet $[\theta] \colon \pi \to e \times Te$, a regular $\mathbf{T}$-graph $\langle\theta\rangle = (b', a')$ on e, and $(\mathrm{id}_e, p) \colon \theta \to \langle\theta\rangle$ is a morphism, where p is the epimorphism in the canonical decomposition of $[\theta]$.

Now we will show that, if $\overline{\theta} = (b, a, i, k)$ is a $\mathbf{T}$-category on e, then we can form a $\mathbf{T}$-preorder $\langle\overline{\theta}\rangle$ over $\langle\theta\rangle$. If (v_1, v_2) is the fibre product of (a, Tb) used in the construction of $\overline{\theta}$, and if (v_1', v_2') is a fibre product of (a', Tb'), let $p' \colon \pi_2 \to \pi_2'$ and $s' \colon \pi_2' \to \pi_2$ be the crochets characterised by the relations

$$\begin{aligned}
v_1' \cdot p' &= p \cdot v_1 \\
v_2' \cdot p' &= Tp \cdot v_2 \\
v_1 \cdot s' &= s \cdot v_1' \\
v_2 \cdot s' &= Ts \cdot v_2'
\end{aligned}$$

where s is a section of p; then $p' \cdot s' = \mathrm{id}_{\pi'_2}$. Set

$$i' = p \cdot i \quad \text{and} \quad k' = p \cdot k \cdot s;$$

by composing on the left with a' and b', we see that $\langle\overline{\theta}\rangle = (b', a', i', k')$ is a $\mathbf{T}$-preorder. We can further show that $(\mathrm{id}_e, p)\colon \overline{\theta} \to \langle\overline{\theta}\rangle$ is a morphism that defines a natural transformation of an adjunction to the inclusion functor $\mathsf{Ord}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T})$. □ *p. 240*

$$\begin{tikzcd}
 & & & T\pi \arrow[dlll, "Ta"'] \arrow[dl, "Tp"] \arrow[dddlll, "Tb"] & & \pi_2 \arrow[ll, "v_2"'] \arrow[dl, "p'"] \arrow[ddll, "v_1"] \\
TTe & & T\pi' \arrow[ll, "Ta'"] \arrow[ur, bend left, "Ts"] \arrow[ddll, "Tb'"'] & & \pi'_2 \arrow[ll, "v'_2"'] \arrow[ur, bend left, "s'"] \arrow[ddll, "v'_1"'] & \\
 & & & \pi \arrow[dl, "p"] \arrow[dlll, "a"'] \arrow[ddlll, "b"] & & \\
Te & & \pi' \arrow[ll, "a'"] \arrow[ur, bend left, "s"] \arrow[dll, "b'"'] & & & \\
e & & & & &
\end{tikzcd}$$

In general, the forgetful functor $\mathsf{Alg}(\mathbf{T}) \to \mathscr{E}$ is not fibrant. Later on, **Proposition 11** resolves the problem of "making" this functor fibrant by universally embedding $\mathsf{Alg}(\mathbf{T})$ into a full subcategory of $\mathsf{Ord}(\mathbf{T})$. For this, we will need to generalise [Ba, Proposition 2.4] by replacing the "category of sets" there with an arbitrary category $\mathscr{E}$, though one that satisfies the following axiom:

(U) There exists a universe such that:

 (a) $\mathrm{Hom}_{\mathscr{E}}$ takes values in this universe
 (b) The set of isomorphism classes of subobjects of an arbitrary object e of $\mathscr{E}$, and the set of isomorphism classes of quotient objects of e, are elements of this universe.
 (c) $\mathscr{E}$ admits projective $\mathscr{A}$-limits for every category $\mathscr{A}$ associated to this universe (i.e. $|\mathscr{A}|$ is an element of this universe and $\mathrm{Hom}_{\mathscr{A}}$ takes values in this universe). *p. 241*

For example, the category Set associated to an arbitrary universe $|\mathsf{Set}|$ satisfies this axiom.

Proposition I.3.10. (Stone–Čech–Barr). *The full inclusion functor* $\mathsf{Alg}(\mathbf{T}) \to \mathsf{Ord}(\mathbf{T})$ *admits an adjoint.*

Proof. Taking into account hypothesis (b) concerning quotients, which allows us to form images of morphisms of $\mathbf{T}$-algebras, the proof is the same as that by Barr, which relies on Freyd's criteria for existence of adjoints (in a form adapted to the terminology of universes, which does not pose any problem). We refer the reader to [Ba]. □

We denote by $m_\theta\colon \theta \to \hat{\theta}$ an **adjunction morphism** associated to a $\mathbf{T}$-preorder θ. Let $\mathsf{Equ}(\mathbf{T})$ be the full subcategory of $\mathsf{Ord}(\mathbf{T})$ whose objects are the $\mathbf{T}$-preorders θ for which m_θ is a cartesian morphism (with respect to the forgetful functor to $\mathcal{E}$). The objects of $\mathsf{Equ}(\mathbf{T})$ are called $\mathbf{T}$-*equivalences.*

Proposition I.3.11. *The forgetful functor* $\mathsf{Equ}(\mathbf{T}) \to \mathcal{E}$ *is fibrant and compatible with the fibration* $\mathsf{Ord}(\mathbf{T}) \to \mathcal{E}$*. For every commutative diagram*

$$\begin{tikzcd} \mathsf{Alg}(\mathbf{T}) \arrow[rr, "X"] \arrow[dr, "U"'] & & \mathcal{X} \arrow[dl, "U'"] \\ & \mathcal{E} & \end{tikzcd}$$

where the forgetful functor of $\mathbf{T}$*-algebras factors through a fibrant functor* $X\colon \mathcal{X} \to \mathcal{E}$*, there exists exactly one (up to equivalence) functor* $\mathsf{Equ}(\mathbf{T}) \to \mathcal{X}$ *that satisfies the following properties:*

(1) The following diagram commutes:

$$\begin{tikzcd} \mathsf{Alg}(\mathbf{T}) \arrow[r] \arrow[dr, "U"'] & \mathsf{Equ}(\mathbf{T}) \arrow[r] \arrow[d] & \mathcal{X} \arrow[dl, "U'"] \\ & \mathcal{E} & \end{tikzcd}$$

(2) The functor sends every morphism that is cartesian with respect to $\mathsf{Equ}(\mathbf{T}) \to \mathcal{E}$ *to a morphism that is cartesian with respect to* $U'\colon \mathcal{X} \to \mathcal{E}$*.*[3]

Proof. Let θ be an $\mathbf{T}$-equivalence, and $f_1\colon e_1 \to |\theta|$ be a morphism in $\mathcal{E}$. If $f\colon f_1^*\theta \to \theta$ is the cartesian morphism such that $|f| = f_1$, we will show that $\theta_1 = f_1^*\theta$ is a $\mathbf{T}$-equivalence, i.e. that m_{θ_1} is cartesian. Let $g\colon \theta_2 \to \hat{\theta}_1$ be a morphism in $\mathsf{Ord}(\mathbf{T})$, and $h_1\colon |\theta_2| \to e_1$ be such that $|m_{\theta_1}| \cdot h_1 = |g|$; we will show that there exists exactly one morphism $h\colon \theta_2 \to \theta_1$ in $\mathsf{Ord}(\mathbf{T})$ such that *p. 242*

$$m_{\theta_1} \cdot h = g \quad \text{and} \quad |h| = h_1.$$

So let $h\colon \theta_2 \to \theta_1$ be such that

$$|h| = h_1 \quad \text{and} \quad m_\theta \cdot f \cdot h = \hat{f} \cdot g$$

(where $\hat{f}$ is the image of f under the adjoint functor of **Proposition I.5.10??**); h exists, since m_θ and f are cartesian, as is their composition. The equality $m_{\theta_1} \cdot h = g$ is

[3] When U is fibrant, **we will refrain from assuming that $\mathsf{Alg}(\mathbf{T}) \simeq \mathsf{Equ}(\mathbf{T})$** (for example, when $\mathbf{T}$ is the identity triple).

a consequence of the faithfulness of the functor $\mathsf{Ord}(\mathbf{T}) \to \mathcal{E}$, as is the uniqueness of h. This proves the first part.

Let $X\colon \mathsf{Alg}(\mathbf{T}) \to \mathcal{X}$ and $U'\colon \mathcal{X} \to \mathcal{E}$ be functors such that $U = U' \cdot X$ is the forgetful functor of $\mathbf{T}$-algebras and U' is fibrant. For all $\theta \in |\mathsf{Equ}(\mathbf{T})|$, let $\hat{X}(\theta)$ be the source (which we can also write as $|m_\theta|^* X(\hat{\theta})$) of a cartesian morphism m in $\mathcal{X}$ with target $X(\hat{\theta})$, such that we have $U'(m) = |m_\theta|$. It is clear that the family $\{\hat{X}(\theta) \mid \theta \in |\mathsf{Equ}(\mathbf{T})|\}$ can be extended in a unique way to a functor $\hat{X}\colon \mathsf{Equ}(\mathbf{T}) \to \mathcal{X}$ that satisfies the required conditions. The equivalence between this functor and every other functor that satisfies the same conditions follows from the fact that two cartesian arrows with the same target and same underlying morphism are related by an invertible. $\square$

As for inductive limits, we content ourselves with the following result, whose elementary proof is left as an exercise.

Proposition I.3.12. *If $\mathcal{E}$ admits inductive $\mathcal{A}$-limits (further to the hypothesis that $\mathcal{E}$ admits finite fibre products), if these limits commute with fibre products, and if T commutes with these limits, then the categories appearing in (∗) admit inductive $\mathcal{A}$-limits, and the functors therein commute with these inductive limits.*

II Pseudo-algebras and monads

II.1 Pseudo-categories

p. 243

This structure slightly generalises that of a bicategory, defined by Bénabou in [Be]; indeed, in bicategories, the families l, r, and s considered below reduce to the families of natural equivalences, which will not always be the case in our examples. (In §III we will see more general structures.)

A *pseudo-category* is a septuple

$$\mathcal{D} = (|\mathcal{D}|, \mathrm{Hom}_{\mathcal{D}}, \iota, \kappa, l, r, s)$$

such that conditions (1) to (8) below are satisfied:

(1) $|\mathcal{D}|$ is a set; its elements are called *objects of* $\mathcal{D}$.

(2) $\mathrm{Hom}_{\mathcal{D}}$ is a family of categories indexed by $|\mathcal{D}|^2$. Saying that $f\colon e \to e'$ is a *morphism in* $\mathcal{D}$ means that

$$e, e' \in |\mathcal{D}| \quad \text{and} \quad f \in |\mathrm{Hom}_{\mathcal{D}}(e', e)|.$$

Saying that $\alpha\colon f \to g\colon e \to e'$, or simply that $\alpha\colon f \to g$, is a *2-morphism in* $\mathcal{D}$ means that $f\colon e \to e'$ and $g\colon e \to e'$ are morphisms in $\mathcal{D}$ and $\alpha\colon f \to g$ is a morphism in $\mathrm{Hom}_{\mathcal{D}}(e', e)$.

(3) ι is a family of functors indexed by $|\mathcal{D}|$, of the form $\iota(e)\colon \mathbf{1} \to \mathrm{Hom}_{\mathcal{D}}(e', e)$ for each $e \in |\mathcal{D}|$, where $\mathbf{1}$ is the category consisting of a single object 0 and the single morphism id_0. We denote by $\mathrm{id}(e)$, or id_e, or simply e, the morphism $\iota(e)(\mathrm{id}_0)\colon e \to e'$.

(4) κ is a family of functors indexed by $|\mathscr{D}|^3$, of the form

$$\kappa(e'',e',e)\colon \operatorname{Hom}_{\mathscr{D}}(e'',e') \times \operatorname{Hom}_{\mathscr{D}}(e',e) \to \operatorname{Hom}_{\mathscr{D}}(e'',e)$$

for each $(e'',e',e) \in |\mathscr{D}|^3$. The image of a pair of morphisms or of 2-morphisms under this functor is called a composition and is denoted by the same symbol, which will generally be the symbol $\circ$ in this paper. This composition is called the *first law of* $\mathscr{D}$, to distinguish it from the composition laws in the $\operatorname{Hom}_{\mathscr{D}}(e',e)$. These latter laws will also be denoted by a unique symbol, called the symbol of the *second law of* $\mathscr{D}$.

p. 244

(5) l and r are two families of natural transformations, each indexed by $|\mathscr{D}|^2$, of the form

$$l(e',e)\colon \mathrm{pr}_2 \to \kappa(e',e',e) \cdot \big(\iota(e') \times \mathrm{id}(\operatorname{Hom}_{\mathscr{D}}(e',e))\big)$$
$$r(e',e)\colon \mathrm{pr}_1 \to \kappa(e',e,e) \cdot \big(\mathrm{id}(\operatorname{Hom}_{\mathscr{D}}(e',e)) \times \iota(e)\big)$$

where

$$\mathrm{pr}_2\colon \mathbf{1} \times \operatorname{Hom}_{\mathscr{D}}(e',e) \to \operatorname{Hom}_{\mathscr{D}}(e',e)$$
$$\mathrm{pr}_1\colon \operatorname{Hom}_{\mathscr{D}}(e',e) \times \mathbf{1} \to \operatorname{Hom}_{\mathscr{D}}(e',e)$$

denote the canonical projections of the products. In other words, for each morphism $f\colon e \to e'$ in $\mathscr{D}$, r and l give "natural" 2-morphisms, denoted in shorthand by

$$l(f)\colon f \to \mathrm{id}_{e'} \circ f \quad \text{and} \quad r(f)\colon f \to f \circ \mathrm{id}_{e'}$$

where $\circ$ is the first law of $\mathscr{D}$.

(6) s is a family of natural transformations, indexed by $|\mathscr{D}|^4$, of the form

$$s(e_4,e_3,e_2,e_1)\colon \kappa_{421} \cdot (\kappa_{432} \times \mathscr{D}_{21}) \to \kappa_{431} \cdot (\mathscr{D}_{43} \times \kappa_{321})$$

for each $(e_4,e_3,e_2,e_1) \in |\mathscr{D}|^4$, where we set

$$\kappa_{ijk} = \kappa(e_i,e_j,e_k) \quad \text{and} \quad \mathscr{D}_{ij} = \operatorname{Hom}_{\mathscr{D}}(e_i,e_j)$$

for $1 \leqslant i,j,k \leqslant 4$. Both these functors have source category $\mathscr{D}_{43} \times \mathscr{D}_{32} \times \mathscr{D}_{21}$ and target category $\mathscr{D}_{41}$. In other words, if $f\colon e_1 \to e_2$, $g\colon e_2 \to e_3$, and $h\colon e_3 \to e_4$ are morphisms in $\mathscr{D}$, then s gives a "natural" 2-morphism, denoted in shorthand by

$$s(h,g,f)\colon (h \circ g) \circ f \to h \circ (g \circ f).$$

We also say that s *shifts parentheses to the right.*

(7) Let $\varepsilon\colon \mathscr{D}_{32} \times \mathscr{D}_{21} \to \mathscr{D}_{32} \times \mathscr{D}_{22} \times \mathscr{D}_{21}$ be the functor given by the composition of the trivial functor $\mathscr{D}_{32} \times \mathscr{D}_{21} \to \mathscr{D}_{32} \times \mathbf{1} \times \mathscr{D}_{21}$ and the functor $\mathscr{D}_{32} \times \iota_2 \times \mathscr{D}_{21}$, using the above notation, as well as setting

$$\begin{aligned} \iota_i &= \iota(e_i) \\ s_{ijkl} &= s(e_i,e_j,e_k,e_l) \\ l_{ij} &= l(e_i,e_j) \\ r_{ij} &= r(e_i,e_j) \end{aligned}$$

where e_i, e_j, e_k, and e_l are objects of $\mathscr{D}$.

For each $(e_1, e_2, e_3) \in |\mathscr{D}|^3$, the following diagram of natural transformations must commute: *p. 245*

$$\begin{tikzcd}
\kappa_{321} \cdot (\kappa_{332} \times \mathscr{D}_{21}) \cdot \varepsilon \arrow[rr, "s_{3221} \cdot \varepsilon"] & & \kappa_{321} \cdot (\mathscr{D}_{32} \times \kappa_{221}) \cdot \varepsilon \\
& \kappa_{321} \arrow[ul, "\kappa_{321} \cdot (r_{32} \times \mathscr{D}_{21})"] \arrow[ur, "\kappa_{321} \cdot (\mathscr{D}_{32} \times l_{21})"'] &
\end{tikzcd}$$

In other words, applying this diagram to a pair of morphisms $f \colon e_1 \to e_2$ and $g \colon e_2 \to e_3$, we should obtain a commutative diagram in the category $\mathrm{Hom}_{\mathscr{D}}(e_3, e_1)$:

$$\begin{tikzcd}
(g \circ \mathrm{id}_e) \circ f \arrow[rr, "s(g, \mathrm{id}_e, f)"] & & g \circ (\mathrm{id}_e \circ f) \\
& g \circ f \arrow[ul, "r(g) \circ f"] \arrow[ur, "g \circ l(f)"'] &
\end{tikzcd}$$

(where we set $e = e_2$).

(8) For all $(e_1, e_2, e_3, e_4, e_5) \in |\mathscr{D}|^5$, we have a commutative diagram

$$\begin{tikzcd}
& \bullet \arrow[dl, "\kappa_{521} \cdot (s_{5432} \times \mathscr{D}_{21})"'] \arrow[ddr, "s_{5321} \cdot (\kappa_{543} \times \mathscr{D}_{32} \times \mathscr{D}_{21})"] & \\
\bullet \arrow[dd, "s_{5421} \cdot (\mathscr{D}_{54} \times \kappa_{432} \times \mathscr{D}_{21})"'] & & \\
& & \bullet \arrow[ddl, "s_{5431} \cdot (\mathscr{D}_{54} \times \mathscr{D}_{43} \times \kappa_{321})"] \\
\bullet \arrow[dr, "\kappa_{541} \cdot (\mathscr{D}_{54} \times s_{4321})"'] & & \\
& \bullet &
\end{tikzcd}$$

(where the notation of the objects is left to the reader) so that "applying" this diagram to morphisms in $\mathscr{D}$

$f \colon e_1 \to e_2$

$g \colon e_2 \to e_3$

$h \colon e_3 \to e_4$

$k \colon e_4 \to e_5$

we obtain a commutative diagram in $\mathrm{Hom}_{\mathcal{D}}(e_5, e_1)$:

$$\begin{tikzcd}
(k\circ(h\circ g))\circ f \arrow[dd, "s(k,h\circ g,f)"'] & ((k\circ h)\circ g)\circ f \arrow[l, "s(k,h,g)\circ f"'] \arrow[d, "s(k\circ h,g,f)"] \\
& (k\circ h)\circ(g\circ f) \arrow[d, "s(k,h,g\circ f)"] \\
k\circ((h\circ g)\circ f) \arrow[r, "k\circ s(h,g,f)"'] & k\circ(h\circ(g\circ f))
\end{tikzcd}$$

Remark. Axioms (7) and (8) are called the “coherence axioms”. If r, l, and s are equivalences (resp. identities) then we recover the notion of bicategory from [Be] (resp. of 2-category). In §III, multicategories will give, as particular cases, both pseudo-categories and the double categories of Ehresmann. *p. 246*

We say that $F\colon \mathcal{D} \to \mathcal{D}'$ is a *pseudo-functor* if $\mathcal{D}$ and $\mathcal{D}'$ are pseudo-categories and F is a quadruple $(|F|, F_1, u, c)$ such that:

(1’) $|F|\colon |\mathcal{D}| \to |\mathcal{D}'|$ is a function.

(2’) F_1 is a family of functors

$$F_1(e', e)\colon\ \mathrm{Hom}_{\mathcal{D}}(e', e) \to \mathrm{Hom}_{\mathcal{D}'}(F(e'), F(e))$$

for each $(e', e) \in |\mathcal{D}|^2$. If $e \in |\mathcal{D}|$, we denote by $F(e)$ its image under $|F|$ whenever there is no risk of confusion. Similarly, if f is a morphism in $\mathcal{D}$ and α a 2-morphism in $\mathcal{D}$, we denote by $F(f)$ and $F(\alpha)$ their images under F_1 (respectively).

(3’) u is a family of natural transformations indexed by $|\mathcal{D}|$, between functors whose source is $\mathbf{1}$; in other words, u can be considered as a family of morphisms in $\mathcal{D}$ of the form

$$u(e)\colon \mathrm{id}_{F(e)} \to F(\mathrm{id}_e)$$

for each $e \in |\mathcal{D}|$.

(4’) c is a family of natural transformations indexed by $|\mathcal{D}|^3$, of the form

$$c(e'', e', e)\colon \kappa\big(F(e''), F(e'), F(e)\big)\cdot\big(F_1(e'', e')\times F_1(e', e)\big) \to F_1(e'', e)\cdot\kappa(e'', e', e).$$

In other words, for each pair of morphism $f\colon e \to e'$ and $g\colon e' \to e''$, we give a “natural” 2-morphism, denoted in shorthand by

$$c(g, f)\colon F(g)\circ F(f) \to F(g\circ f).$$

(5') For each $f\colon e \to e'$ we have two commutative diagrams:

$$\begin{tikzcd} F(f) \arrow[r, "F(r(f))"] \arrow[d, "r(F(f))"'] & F(f \circ \mathrm{id}_e) \\ F(f) \circ \mathrm{id}_{F(e)} \arrow[r, "F(f)\circ u(e)"'] & F(f) \circ F(\mathrm{id}_e) \arrow[u, "c(f,\mathrm{id}_e)"'] \end{tikzcd} \qquad \begin{tikzcd} F(\mathrm{id}_{e'} \circ f) & F(f) \arrow[l, "F(l(f))"'] \arrow[d, "l(F(f))"] \\ F(\mathrm{id}_{e'}) \circ F(f) \arrow[u, "c(\mathrm{id}_{e'},f)"] & \mathrm{id}_{F(e')} \circ F(f) \arrow[l, "u(e')\circ F(f)"] \end{tikzcd}$$

p. 247

(6') For any triple $f\colon e \to e'$, $g\colon e' \to e''$, and $h\colon e'' \to e'''$ in $\mathscr{D}$, we have a commutative diagram

$$\begin{tikzcd} F(h) \circ (F(g) \circ F(f)) \arrow[d, "F(h)\circ c(g,f)"'] & (F(h) \circ F(g)) \circ F(f) \arrow[l, "s(F(h),F(g),F(f))"'] \arrow[d, "c(h,g)\circ F(f)"] \\ F(h) \circ F(g \circ f) \arrow[d, "c(h,g\circ f)"'] & F(h \circ g) \circ F(f) \arrow[d, "c(h\circ g,f)"] \\ F(h \circ (g \circ f)) & F((h \circ g) \circ f) \arrow[l, "F(s(h,g,f))"] \end{tikzcd}$$

All of these definitions are based on those of bicategories of Bénabou [Be] (we note that we have, however, reversed the direction of the arrows r and l). We define the composition of pseudo-functors in a manner analogous to that of bifunctors [Be]. We could define the notions of natural pseudo-transformation and even of morphisms between these. This would allow us to define the notation of "pseudo-triple" on a pseudo-category, which we will not do. We simply note that **Proposition 3??** later on would give **T**-categories a particularly agreeable **meaning**.

Remarks. —

(1) These definitions of pseudo-category and pseudo-functor should be taken mainly as provisionary. Indeed, we have simply imitated the definition of bicategories of Bénabou [Be], which itself was inspired by the definition of monoidal categories of Mac Lane; but nothing tells us that there are no missing axioms that we would need in order to obtain the "best" possible structure, nor that the chosen direction for the coherence morphisms — suggested by the example of **T**-spans that follows — is the "good" one. We hope to soon carry out a more serious study of this structure.

(2) In practical examples, not only are the two morphisms $(f\circ g)\circ h$ and $f\circ(g\circ h)$ (where f, g, and h are "consecutive" morphisms in $\mathscr{D}$) not equal, but they are related to a third morphism, denoted $f \circ g \circ h$, and, more generally, we have "compositions" of length $n > 2$ that are different to the compositions obtained by the grouping of the factors. This is the case, for example, when we consider the category Set p. 248

of sets associated to a universe, endowed with the cartesian product as a first law. We wait until the notion of multicategory defined later on (**??**) allows us to unify these differing notions.

II.2 The pseudo-category of T-spans

Let $\mathscr{E}$ be a category endowed with a triple $\mathbf{T}$, admitting finite fibre products and endowed with a canonical choice of these fibre products. We will associate to this data a pseudo-category $\mathsf{Sp}(\mathbf{T})$ constructed in the following way:

(1) $|\mathsf{Sp}(\mathbf{T})| = |\mathscr{E}|$

(2) For every $e, e \in |\mathscr{E}|$, let $\mathrm{Hom}_{\mathsf{Sp}(\mathbf{T})}(e', e)$ be the category whose objects are the $\mathbf{T}$-spans $\theta\colon e \to e'$, i.e. the pairs (b, a), where

$$b\colon \pi \to Te' \quad \text{and} \quad a\colon \pi \to e$$

are morphisms in $\mathscr{E}$ with the same source. If $\theta'\colon e \to e'$ is another $\mathbf{T}$-span, where $\theta' = (b', a')$ with $b'\colon \pi' \to Te'$ and $a'\colon \pi' \to e$, then a morphism $m\colon \theta \to \theta'$ in $\mathrm{Hom}_{\mathsf{Sp}(\mathbf{T})}(e', e)$ (and thus a 2-morphism in $\mathsf{Sp}(\mathbf{T})$) is defined by a morphism $m\colon \pi \to \pi'$ in $\mathscr{E}$ satisfying the following conditions:

$$b' \cdot m = b \quad \text{and} \quad a' \cdot m = a.$$

$$\begin{tikzcd}
& \pi \arrow[dl, "b"'] \arrow[dr, "a"] \arrow[dd, "m"] & \\
Te' & & e \\
& \pi' \arrow[ul, "b'"] \arrow[ur, "a'"'] &
\end{tikzcd}$$

The evident composition law on $\mathrm{Hom}_{\mathsf{Sp}(\mathbf{T})}(e', e)$ is denoted by $\cdot$ just as for $\mathscr{E}$.

(3) For each $e \in |\mathscr{E}|$, we define a $\mathbf{T}$-span $(Ie, \mathrm{id}_e)\colon e \to Te$, which we also denote by id_e if the context clearly specifies that it is a $\mathbf{T}$-span.

(4) If $\theta\colon e \to e'$ and $\theta'\colon e' \to e''$ are $\mathbf{T}$-spans, we define the first composition law of $\mathsf{Sp}(\mathbf{T})$ by setting

$$\theta' \circ \theta = (Ke'' \cdot Tb' \cdot v_1, a \cdot v_2)$$

where (v_1, v_2) is a canonical fibre product of (Ta', b). The composition of 2-morphisms is then obtained thanks to the property of fibre products. *p. 249*

TO-DO diagram

Proposition II.2.13. *If $\theta\colon e \to e'$, $\theta'\colon e' \to e''$, and $\theta''\colon e'' \to e'''$ are $\mathbf{T}$-spans, then we can construct 2-morphisms in $\mathsf{Sp}(\mathbf{T})$*

$$\mathrm{id}_{e'} \circ \theta \xleftarrow{l(\theta)} \theta \xrightarrow{r\theta} \theta \circ \mathrm{id}_e$$

$$\theta'' \circ (\theta' \circ \theta) \xrightarrow{s(\theta'', \theta', \theta)} (\theta'' \circ \theta') \circ \theta$$

(where $r(\theta)$ is furthermore an isomorphism), which will finish the construction of $\mathsf{Sp}(\mathbf{T})$ *as a pseudo-category.*

Proof. Set $\theta = (b, a)$; form the canonical fibre products $(\overline{v}'_1, \overline{v}'_2)$ of $(\mathrm{id}_{Te'}, b)$ and $(\overline{v}_1, \overline{v}_2)$ of (Ta, Ie). Since $\mathrm{id}_{Te'}$ is an isomorphism, so too is $\overline{v}'_2$; let ${\overline{v}'_2}^{-1}$ be its inverse. The relations

$$(Ke' \cdot TIe' \cdot \overline{v}'_1) \cdot {\overline{v}'_2}^{-1} = \overline{v}'_1 \cdot {\overline{v}'_2}^{-1} = b$$

(since $b \cdot \overline{v}'_2 = \overline{v}'_1$) and

$$(a \cdot \overline{v}'_2) \cdot {\overline{v}'_2}^{-1} = a$$

allow us to set

$$l(\theta) = {\overline{v}'_2}^{-1} \colon \theta \to \mathrm{id}_{e'} \circ \theta.$$

The relation $Ta \cdot I\pi = Ie \cdot a$ gives a crochet $c \colon \pi \to \pi_2$ such that

$$\overline{v}_1 \cdot c = I\pi \quad \text{and} \quad \overline{v}_2 \cdot c = a.$$

The relations

$$\begin{aligned} (Ke' \cdot Tb \cdot \overline{v}_1) \cdot c &= b \\ (\mathrm{id}_e \cdot \overline{v}_2) \cdot c &= a \end{aligned}$$

allow us to set

$$r(\theta) = c \colon \theta \to \theta \circ \mathrm{id}_e.$$

Set $\theta' = (b', a')$ and $\theta'' = (b'', a'')$; the construction of the composites $(\theta'' \circ \theta') \circ \theta$ and $\theta'' \circ (\theta' \circ \theta)$ leads to forming fibre products that we can follow, for notation, in the diagrams below. *p. 250*

Firstly, $(\theta'' \circ \theta') \circ \theta \colon e \to e'''$ is defined by the pair

$$\big(Ke''' \cdot T(Ke''' \cdot Tb'' \cdot v'_1) \cdot w_1, a \cdot v_2 \cdot w_2\big).$$

TO-DO diagram

(Note that, if (v_1, v_2) is the canonical fibre product of (Ta', b), then (w_1, w_2) is actually chosen so that $(w_1, v_2 \cdot w_2)$ be the canonical fibre product of $(Tv'_2 \cdot Ta', b)$, which is evidently possible.)

TO-DO diagram *p. 251*

Secondly, $\theta'' \circ (\theta' \circ \theta) \colon e \to e'''$ is defined by the pair

$$(Ke''' \cdot Tb'' \cdot W'_1, a \cdot v_2 \cdot w'_2).$$

By comparing the relations

$$(Ke'' \cdot Tb' \cdot v_1) \cdot w_2 = Ke'' \cdot TTa'' \cdot Tv'_1 \cdot w_1 = Ta'' \cdot (K\pi'' \cdot Tv'_1 \cdot w_1)$$

with the fibre product (w'_1, w'_2) of $(Ke''{\cdot}Tb'{\cdot}v_1, Ta'')$, we obtain a crochet $s = s(\theta'', \theta', \theta)\colon \hat{\pi} \to \hat{\pi}'$ characterised by the relations

$$\begin{aligned} w'_2 \cdot s &= w_2 \\ w'_1 \cdot s &= K\pi'' \cdot Tv'_1 \cdot w_1 \end{aligned}$$

and so s is the desired associativity morphism, since

$$\begin{aligned} Ke''' \cdot Tb'' \cdot w'_1 \cdot s &= Ke''' \cdot Tb'' \cdot K\pi'' \cdot Tv'_1 \cdot w_1 \\ &= Ke''' \cdot KTe''' \cdot TTb'' \cdot Tv'_1 \cdot w_1 \\ &= Ke''' \cdot TKe''' \cdot TTb'' \cdot Tv'_1 \cdot w_1 \end{aligned}$$

and

$$a \cdot v_w \cdot w'_2 \cdot s = a \cdot v_2 \cdot w_2$$

which shows that $s\colon (\theta'' \circ \theta') \circ \theta \to \theta'' \circ (\theta' \circ \theta)$ defines a morphism in $\mathsf{Sp}(\mathbf{T})$.

We now show the coherence axiom (8). Let

$$\theta_i = (b_i, a_i)\colon e_i \to e_{i+1}$$

(for $1 \leqslant i \leqslant 4$) be a $\mathbf{T}$-span. We set

$$\begin{aligned} p &= s(\theta_4, \theta_3, \theta_2) \circ \theta_1 \\ p' &= s(\theta_4, \theta_3 \circ \theta_2, \theta_1) \\ p'' &= \theta_4 \circ s(\theta_3, \theta_2, \theta_1) \\ q &= s(\theta_4 \circ \theta_3, \theta_2, \theta_1) \\ q' &= s(\theta_4, \theta_3, \theta_2 \circ \theta_1). \end{aligned}$$

We need to show the relation $q' \cdot q = p'' \cdot p' \cdot p$. To simplify notation, we will use the same notation for a morphism between $\mathbf{T}$-spans (i.e. a 2-morphism in $\mathsf{Sp}(\mathbf{T})$) and the underlying morphism in $\mathcal{E}$ that defines it, as long as the context ensures that no confusion may arise. Finally, we note that the fibre products are not always chosen so as to be canonical, but instead so as to make the other fibre products canonical.

Construction of $((\theta_4 \circ \theta_3) \circ \theta_2) \circ \theta_1$. Let (u_i, u'_i) be the canonical fibre product of (Ta_{i+1}, b_i) for $1 \leqslant i \leqslant 3$, so that $\theta_{i+1} {\circ} \theta_i\colon e_i \to e_{i+2}$ is defined by $(Ke_{i+1}{\cdot}Tb_{i+1}{\cdot}u_i, a_i{\cdot}u'_i)$ (for $1 \leqslant i \leqslant 3$). Let (v_i, v'_i) be a fibre product of (Tu'_{i+1}, u_i) for $1 \leqslant i \leqslant 2$, so that $(v_i, u'_i \cdot v'_i)$ is the fibre product of $(Ta_{i+1} \cdot Tu_{i+1}, b_i)$ and $(\theta_{i+2} \circ \theta_{i+1}) \circ \theta_i$ is defined by $(Ke_{i+3} \cdot T(Ke_{i+3} \cdot Tb_{i+2} \cdot u_{i+1}) \cdot v_i, a_i \cdot u'_i \cdot v'_i)$ for $1 \leqslant i \leqslant 2$. Let (w_1, w'_1) be the fibre product of (Tv'_2, v_1), and π_i and π'_i the source of a_i and u_i (respectively). *p. 252*

Construction of $\theta_4 \circ (\theta_3 \circ (\theta_2 \circ \theta_1))$. Let (m_1, m'_1) be the canonical fibre product of $(Ta_3, Ke_3 \cdot Tb_2 \cdot u_1)$, so that $\theta_3 \circ (\theta_2 \circ \theta_1)$ is defined by $(Ke_4 \cdot Tb_3 \cdot m_1, a_1 \cdot u'_1 \cdot m'_1)$. Let (o, o') be the canonical fibre product of $(Ta_4, Ke_4 \cdot Tb_3 \cdot m_1)$, so that $\theta_4 \circ (\theta_3 \circ (\theta_2 \circ \theta_1))$ is defined by $(Ke_5 \cdot Tb_4 \cdot o, a_1 \cdot u'_1 \cdot m'_1 \cdot o')$.

Construction of $(\theta_4 \circ \theta_3) \circ (\theta_2 \circ \theta_1)$. Let (r, r') be a fibre product of (Tu'_3, m_1), so that $(\theta_4 \circ \theta_3) \circ (\theta_2 \circ \theta_1)$ is defined by

$$(Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot u_3) \cdot r, a_1 \cdot u'_1 \cdot m'_1 \cdot r').$$

Construction of q. The relation

$$T(a_3 \cdot u_3') \cdot K\pi_3' \cdot Tv_2 \cdot w_1 = (Ke_3 \cdot Tb_2 \cdot u_1) \cdot v_1' \cdot w_1'$$

combined with the fibre product $(r, m_1' \cdot r')$ of $(T(a_3 \cdot u_3'), Ke_3 \cdot Tb_2 \cdot u_1)$ gives a **crochet** q (a morphism in $\mathcal{E}$) characterised by

$$\begin{aligned} r \cdot q &= K\pi_3' \cdot Tv_2 \cdot w_1 \\ m_1' \cdot r' \cdot q &= v_1' \cdot w_1'. \end{aligned} \tag{1}$$

Then

$$q\colon ((\theta_4 \circ \theta_3) \circ \theta_2)\theta_1 \to (\theta_4 \circ \theta_3) \circ (\theta_2 \circ \theta_1)$$

is a morphism in $\mathsf{Sp}(\mathbf{T})$. Indeed, by (1),

$$\begin{aligned} Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot u_3) \cdot r \cdot q &= Ke_5 \cdot TKe_5 \cdot TTb_4 \cdot Tu_3 \cdot K\pi_3' \cdot Tv_2 \cdot w_1 \\ &= Ke_5 \cdot TKe_5 \cdot KTTe_5 \cdot TTTb_4 \cdot TTu_3 \cdot Tv_2 \cdot w_1 \\ &= Ke_5 \cdot T(Ke_5 \cdot KTe_5 \cdot TTb_4 \cdot Tu_3 \cdot v_2) \cdot w_1 \\ &= Ke_5 \cdot T(Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot u_3) \cdot v_2) \cdot w_1 \end{aligned}$$

and

$$a_1 \cdot u_1' \cdot m_1' \cdot r' \cdot q = a_1 \cdot u_1' \cdot v_1' \cdot w_1'.$$

Construction of q'. The relations

$$\begin{aligned} Ta_4 \cdot (K\pi_4 \cdot Tu_3 \cdot r) &= Ke_4 \cdot Tb_3 \cdot Tu_3' \cdot r \\ &= (Ke_4 \cdot Tb_3 \cdot m_1) \cdot r' \end{aligned}$$

combined with the fibre product (o, o') of $(Ta_4, Ke_4 \cdot Tb_3 \cdot m_1)$ give a **crochet** q' (a morphism in $\mathcal{E}$) characterised by *p. 253*

$$\begin{aligned} o \cdot q' &= K\pi_4 \cdot Tu_3 \cdot r \\ o' \cdot q' &= r'. \end{aligned} \tag{2}$$

Then

$$q'\colon (\theta_4 \circ \theta_3) \circ (\theta_2 \circ \theta_1) \to \theta_4 \circ (\theta_3 \circ (\theta_2 \circ \theta_1))$$

is a morphism in $\mathsf{Sp}(\mathbf{T})$. Indeed, by (2),

$$\begin{aligned} (Ke_5 \cdot Tb_4 \cdot o) \cdot q' &= Ke_5 \cdot Tb_4 \cdot K\pi_4 \cdot Tu_3 \cdot r \\ &= Ke_5 \cdot KTe_5 \cdot TTb_4 \cdot Tu_3 \cdot r \\ &= Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot u_3) \cdot r \end{aligned}$$

and

$$a_1 \cdot u_1' \cdot m_1' \cdot o' \cdot q' = a_1 \cdot u_1' \cdot m_1' \cdot r'.$$

Construction of $(\theta_4 \circ (\theta_3 \circ \theta_2)) \circ \theta_1$. Let (m_2, m_2') be the canonical fibre product of $(Ta_4, Ke_4 \cdot Tb_3 \cdot u_2)$, so that $\theta_4 \circ (\theta_3 \circ \theta_2)$ is defined by $(Ke_5 \cdot Tb_4 \cdot m_2, a_2 \cdot u_2' \cdot m_2')$, and let (t, t') be a fibre product of (Tm_2', v_1), so that $(\theta_4 \circ (\theta_3 \circ \theta_2)) \circ \theta_1$ is defined by

$$(Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot m_2) \cdot t, a_1 \cdot u_1' \cdot v_1' \cdot t')$$

(in other words, $(t, u_1' \cdot v_1' \cdot t')$ is the canonical fibre product of $(Ta_2 \cdot Tu_2' \cdot Tm_2', b_1)$).

Construction of p. The relations

$$\begin{aligned} Ta_4 \cdot (K\pi_4 \cdot Tu_3 \cdot v_2) &= Ke_4 \cdot Tb_3 \cdot Tu_3' \cdot v_2 \\ &= (Ke_4 \cdot Tb_3 \cdot u_2) \cdot v_2' \end{aligned}$$

combined with the fibre product (m_2, m_2') of $(Ta_4, Ke_4 \cdot Tb_3 \cdot u_2)$ give a crochet d_2 characterised by

$$\begin{aligned} m_2 \cdot d_2 &= K\pi_4 \cdot Tu_3 \cdot v_2 \\ m_2' \cdot d_2 v_2' \end{aligned} \tag{3}$$

(this is in fact the morphism in $\mathcal{E}$ that defines $s(\theta_4, \theta_3, \theta_2)$). Now the relations

$$\begin{aligned} Tm_2' \cdot Td_2 \cdot w_1 &= T(m_2' \cdot d_2) \cdot w_1 \\ &= Tv_2' \cdot w_1 \\ &= v_1 \cdot w_1' \end{aligned}$$

combined with the fibre product (t, t') give a crochet p characterised by

$$\begin{aligned} t \cdot p &= Td_2 \cdot w_1 \\ t' \cdot p &= w_1'. \end{aligned} \tag{4}$$

Then p. 254

$$p \colon ((\theta_4 \circ \theta_3) \circ \theta_2) \circ \theta_1 \to (\theta_4 \circ (\theta_3 \circ \theta_2)) \circ \theta_1$$

is a morphism in $\mathsf{Sp}(\mathbf{T})$. Indeed, by (3),

$$\begin{aligned} Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot m_2) \cdot t \cdot p &= Ke_5 \cdot TKe_4 \cdot TTb_4 \cdot (Tm_2 \cdot Td_2) \cdot w_1 \\ &\underset{(3)}{=} Ke_5 \cdot TKe_5 \cdot TTb_4 \cdot T(K\pi_4 \cdot Tu_3 \cdot v_2) \cdot w_1 \\ &= Ke_5 \cdot TKe_5 \cdot TKTe_5 \cdot TTTb_4 \cdot TTu_3 \cdot Tv_2 \cdot w_1 \\ &= Ke_5 \cdot T(Ke_5 \cdot KTe_5 \cdot TTb_4 \cdot Tu_3 \cdot v_2) \cdot w_1 \\ &= Ke_5 \cdot T(Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot u_3) \cdot v_2) \cdot w_1 \end{aligned}$$

and, by (4),

$$a_1 \cdot u_1' \cdot v_1' \cdot t' \cdot p = a_1 \cdot u_1' \cdot v_1' \cdot w_1'.$$

Construction of p'. The relations

$$\begin{aligned} Ta_4 \cdot (K\pi_4 \cdot Tm_2 \cdot t) &= Ke_4 \cdot TTa_4 \cdot Tm_2 \cdot t \\ &= Ke_4 \cdot T(Ke_4 \cdot Tb_3 \cdot u_2) \cdot Tm_2' \cdot t \\ &= (Ke_4 \cdot T(Ke_4 \cdot Tb_3 \cdot u_2) \cdot v_1) \cdot t' \end{aligned}$$

combined with the fibre product (n, n') of $(Ta_4, Ke_4 \cdot T(Ke_4 \cdot Tb_3 \cdot u_2) \cdot v_1)$ give a crochet p characterised by

$$\begin{aligned} n \cdot p' &= K\pi_4 \cdot Tm_2 \cdot t \\ n' \cdot p' &= t'. \end{aligned} \tag{5}$$

Then

$$p' \colon (\theta_4 \circ (\theta_3 \circ \theta_2)) \circ \theta_1 \to \theta_4 \circ ((\theta_3 \circ \theta_2) \circ \theta_1)$$

is a morphism in $\mathsf{Sp}(\mathbf{T})$. Indeed, by (5),

$$\begin{aligned} Ke_5 \cdot Tb_4 \cdot n \cdot p' &\underset{(5)}{=} Ke_5 \cdot Tb_4 \cdot K\pi_4 \cdot Tm_2 \cdot t \\ &= Ke_5 \cdot KTe_5 \cdot TTb_4 \cdot Tm_2 \cdot t \\ &= Ke_5 \cdot T(Ke_5 \cdot Tb_4 \cdot m_2) \cdot t \end{aligned}$$

and

$$a_1 \cdot u_1' \cdot v_1' \cdot n' \cdot p' = a_1 \cdot u_1' \cdot v_1' \cdot t'.$$

Construction of p''. The relation

$$Ta_3 \cdot K\pi_3 \cdot Tu_2 \cdot v_1 = Ke_3 \cdot Tb_2 \cdot u_1 \cdot v_1'$$

gives a crochet d_1 characterised by

$$\begin{aligned} m_1 \cdot d_1 &= K\pi_3 \cdot Tu_2 \cdot v_1 \\ m_1' \cdot d_1 &= v_1' \end{aligned} \tag{6}$$

(this is in fact the morphism in $\mathscr{E}$ that defines $s(\theta_3, \theta_2, \theta_1)$). *p. 255*

Now the relations

$$\begin{aligned} Ta_4 \cdot n &= Ke_4 \cdot KTe_4 \cdot TTb_3 \cdot Tu_2 \cdot v_1 \cdot n' \\ &= Ke_4 \cdot Tb_3 \cdot K\pi_3 \cdot Tu_2 \cdot v_1 \cdot n' \\ &= (Ke_4 \cdot Tb_3 \cdot m_1) \cdot d_1 \cdot n' \end{aligned}$$

combined with the fibre product (o, o') of $(Ta_4, Ke_4 \cdot Tb_3 \cdot m_1)$ give a crochet characterised by

$$\begin{aligned} o \cdot p'' &= n \\ o' \cdot p'' &= d_1 \cdot n'. \end{aligned} \tag{7}$$

Then

$$p'' \colon \theta_4 \circ ((\theta_3 \circ \theta_2) \circ \theta_1) \to \theta_4 \circ (\theta_3 \circ (\theta_1 \circ \theta_1))$$

is a morphism in $\mathsf{Sp}(\mathbf{T})$, since

$$Ke_5 \cdot Tb_4 \cdot o \cdot p'' = Ke_5 \cdot Tb_4 \cdot n$$

and, by (6),

$$\begin{aligned} a_1 \cdot u_1' \cdot m_1' \cdot o' \cdot p'' &= a_1 \cdot u_1' \cdot m_1' \cdot d_1 \cdot n' \\ &\underset{(6)}{=} a_1 \cdot u_1' \cdot v_1' \cdot n'. \end{aligned}$$

Construction of the "deus ex machina" x. The relations

$$\begin{aligned} Ta_4 \cdot (K\pi_4 \cdot KT\pi_4 \cdot TTu_3 \cdot Tv_2 \cdot w_1) &= Ke_4 \cdot KTe_4 \cdot TTb_3 \cdot Tu_2 \cdot v_1 \cdot w_1' \\ &= (Ke_4 \cdot T(Ke_4 \cdot Tb_3 \cdot u_2) \cdot v_1) \cdot w_1' \\ &\underset{(6)}{=} (Ke_4 \cdot Tb_3 \cdot m_1) \cdot d_1 \cdot w_1' \end{aligned}$$

combined with the fibre product (o, o') give a crochet x such that

$$\begin{aligned} o \cdot x &= K\pi_4 \cdot KT\pi_4 \cdot TTu_3 \cdot Tv_2 \cdot w_1 \\ o' \cdot x &= d_1 \cdot w_1'. \end{aligned} \tag{8}$$

It is the uniqueness property of this crochet that will allow us to complete the proot; indeed, it suffices to note that if in (8) we replace x by $q' \cdot q$ or $p'' \cdot p' \cdot p$ then these relations still hold true. Indeed,

$$\begin{aligned} o \cdot q' \cdot q &\underset{(2)}{=} K\pi_4 \cdot Tu_3 \cdot r \cdot q \\ &= K\pi_4 \cdot Tu_3 \cdot K\pi_3' \cdot Tv_2 \cdot w_1 \\ &= K\pi_4 \cdot KT\pi_4 \cdot TTu_3 \cdot Tv_2 \cdot w_1 \end{aligned}$$

and

$$o' \cdot q' \cdot q \underset{(2)}{=} r' \cdot q = d_1 \cdot w_1'$$

where the last equality is proven by using the universal property of the fibre product, since, by definition of the fibre product (r, r'), we have that

$$\begin{aligned} m_1 \cdot r' \cdot q &= Tu_3' \cdot r \cdot q \\ &\underset{(1)}{=} Tu_3' \cdot K\pi_3' \cdot Tv_2 \cdot w_1 \\ &= K\pi_3 \cdot Tu_2 \cdot v_1 \cdot w_1' \\ &\underset{(6)}{=} m_1 \cdot d_1 \cdot w_1' \end{aligned}$$

and p. 256

$$m' \cdot r' \cdot q \underset{(1)}{=} v' \cdot w_1' \underset{(6)}{=} m_1' \cdot d_1 \cdot w_1'.$$

Now we replace x by $p'' \cdot p' \cdot p$ in (8), which gives

$$\begin{aligned} o \cdot p'' \cdot p' \cdot p &\underset{(7)}{=} n \cdot p' \cdot p \\ &\underset{(5)}{=} K\pi_4 \cdot Tm_2 \cdot t \cdot p \\ &\underset{(4)}{=} K\pi_4 \cdot T(K\pi_4 \cdot Tu_3 \cdot v_2) \cdot w_1 \\ &= K\pi_4 \cdot KT\pi_4 \cdot TTu_3 \cdot Tv_2 \cdot w_1 \end{aligned}$$

and

$$\begin{aligned} o' \cdot p'' \cdot p' \cdot p &\underset{(7)}{=} d_1 \cdot n' \cdot p' \cdot p \\ &\underset{(5)}{=} d_1 \cdot t_1' \cdot p \\ &\underset{(4)}{=} d_1 \cdot w_1'. \end{aligned}$$

This finishes the proof of the coherence axiom (8). The proof of the coherence axiom (9) is based on the same method. □

Remark. We could have proven a more general result, and more quickly, by systematically using pseudo-triples. $\mathsf{Sp}(\mathbf{T})$ would then have appeared as the "Kleisli" pseudo-category of the pseudo-triple induced by $\mathbf{T}$ on $\mathsf{Sp}(\mathcal{E})$, where $\mathsf{Sp}(\mathcal{E}) = \mathsf{Sp}((\mathrm{id}_{\mathcal{E}}, \mathrm{id}_{\mathcal{E}}, \mathrm{id}_{\mathcal{E}}))$. We canonically obtain a pseudo-functor from the Kleisli category $\mathsf{Kl}(\mathbf{T})$ of the triple $\mathbf{T}$ to the pseudo-category $\mathsf{Sp}(\mathbf{T})$ by sending a morphism $f\colon e \to e'$ in $\mathsf{Kl}(\mathbf{T})$ (i.e. $f\colon e \to Te$ in $\mathcal{E}$) to the $\mathbf{T}$-span

$$(f, \mathrm{id}_e)\colon e \to e'.$$

We say that a natural transformation $m\colon F \to F'$, where $F, F'\colon \mathcal{C} \to \mathcal{C}'$ are functors, is *cartesian* if, for every morphism $f\colon e \to e'$ in $\mathcal{C}$, the pair $(F(f), m(e))$ is a fibre product of $(m(e'), F'(f))$. We say that $\mathbf{T} = (T, I, K)$ is a *cartesian triple* if the following two conditions are satisfied:

(a) E admits finite fibre products, and T commutes with them (which is the case as soon as I or K are cartesian).

(b) I and K are cartesian natural transformations.

We will see examples of cartesian triples in §III; here is a useful characterisation:

Proposition II.2.14. *For $\mathsf{Sp}(\mathbf{T})$ to be a bicategory (i.e. for r, l, and s to define natural equivalences), it is necessary and sufficient that $\mathbf{T}$ be cartesian.*

p. 257

Proof. We use the notation of **Proposition 1**. For the construction of $r(\theta)$, it is clear that condition (c) implies that the crochet between two fibre products is invertible, and conversely if it is invertible then (c) is satisfied. We already know that $l(\theta)$ is always invertible For the construction of $s(\theta'', \theta', \theta)$, **Figure 1** clearly shows that if (a) and (b) are satisfied then $(K\pi'' \cdot Tv_1 \cdot w_1, w_2)$ is a fibre product of $(Ta'', Ke'' \cdot Tb' \cdot v_1)$ and, since s is the crochet arising from the canonical fibre product (w_1, w_2), it is invertible. The converse can be proven by considering degenerate cases of this situation. □

II.3 Two interpretations of T-categories

We will give two interpretations, somehow dual to one another, of $\mathbf{T}$-categories.

A *monad* (or *monoid*) in a pseudo-category $\mathcal{D}$ is a pseudo-functor $\mu\colon \mathbf{1} \to \mathcal{D}$, where we denote by $\mathbf{1}$ the pseudo-category consisting of a single object 0, a single morphism, and a single 2-morphism (this entirely determines the structure). We denote by $\circ$ the first law in $\mathscr{D}$ and by $\cdot$ the second. If $e = \mu(0)$ and $\theta = \mu(\mathrm{id}_0)$, then the data of μ is equivalent to that of two 2-morphisms in $\mathscr{D}$, namely

$$i\colon \mathrm{id}_e \to \theta \quad \text{and} \quad k\colon \theta \circ \theta \to \theta$$

(which correspond to the families u and c in the definition of pseudo-functors) that make the following diagrams commute:

$$\begin{tikzcd} \theta \arrow[r, "\mathrm{id}(\theta)"] \arrow[d, "r(\theta)"'] & \theta \\ \theta \circ \mathrm{id}_e \arrow[r, "\theta\circ i"'] & \theta \circ \theta \arrow[u, "k"'] \end{tikzcd} \qquad \begin{tikzcd} \theta & \theta \arrow[l, "\mathrm{id}(\theta)"'] \arrow[d, "l(\theta)"] \\ \theta \circ \theta \arrow[u, "k"] & \mathrm{id}_e \circ \theta \arrow[l, "i\circ\theta"] \end{tikzcd} \qquad \begin{tikzcd} \theta \circ (\theta \circ \theta) \arrow[d, "\theta\circ k"'] & (\theta \circ \theta) \circ \theta \arrow[l, "s(\theta,\theta,\theta)"'] \arrow[d, "k\circ\theta"] \\ \theta \circ \theta \arrow[d, "k"'] & \theta \circ \theta \arrow[d, "k"] \\ \theta & \theta \arrow[l, "\mathrm{id}(\theta)"] \end{tikzcd}$$

(which correspond to axioms (5') and (6') of §II.1). *p. 258*

Let $\mathscr{E}$ be a category endowed with a triple $\mathbf{T}$, admitting finite fibre products and endowed with a canonical choice of these fibre products.

Proposition II.3.15. *The data of a $\mathbf{T}$-category is equivalent to that of a monad in the pseudo-category $\mathsf{Sp}(\mathbf{T})$ on the same object.*

Proof. If $\mu\colon \mathbf{1} \to \mathsf{Sp}(\mathbf{T})$ is a monad, then the $\mathbf{T}$-span $\theta\colon e \to e$ is defined by a pair (a, b) of morphisms in $\mathscr{E}$ of the form

$$a\colon \pi \to Te \quad \text{and} \quad b\colon \pi \to e.$$

We will construct the 2-morphism

$$s(\theta,\theta,\theta)\colon (\theta \circ \theta) \circ \theta \to \theta \circ (\theta \circ \theta)$$

in $\mathsf{Sp}(\mathbf{T})$. For this, let (v_2, v_1) be the canonical fibre product of (Tb, a), let (w_2, w_1) be the fibre product of (Tv_1, v_2) that makes the fibre product $(w_2, v_1 \cdot w_1)$ of $(Tb \cdot Tv_1, a)$ canonical, and let (w_2', w_1') be the canonical fibre product of $(Tb, Ke \cdot Ta \cdot v_2)$; then $s(\theta,\theta,\theta)$ is defined by the morphism s in $\mathscr{E}$ characterised by

$$\begin{aligned} w_1' \cdot s &= w_1 \\ w_2' \cdot s &= K\pi \cdot Tv_2 \cdot w_2 \end{aligned} \tag{1}$$

which follows from combining the relation

$$Tb \cdot K\pi \cdot Tv_2 \cdot w_2 = Ke \cdot Ta \cdot v_2 \cdot w_1$$

with the fibre product (w_2', w_1').

We now construct the 2-morphisms

$$l(\theta)\colon \theta \to \mathrm{id}_e \circ \theta \quad \text{and} \quad r(\theta)\colon \theta \to \theta \circ \mathrm{id}_e$$

in $\mathsf{Sp}(\mathbf{T})$, defined by the morphisms l and r of $\mathcal{E}$ characterised by the relations

$$\begin{aligned} u'_2 \cdot l &= a \\ u'_1 \cdot l &= \mathrm{id}_\pi \\ u_2 \cdot r &= I\pi \\ u_1 \cdot r &= b \end{aligned} \tag{2}$$

where (u_2, u_1) is the canonical fibre product of (Tb, Ie), and (u'_2, u'_1) that of (id_{Te}, a). In the rest of this proof, we denote by the same letter x a 2-morphism $x\colon e \to e''$ in $\mathsf{Sp}(\mathbf{T})$ and the underlying morphism $x\colon e \to e'$ in $\mathcal{E}$; i and k have the same meaning as above (the families u and c). We will show that $\theta^* = (b, a, i, k)$ is a (non-canonical; see the definition in §II.4) $\mathbf{T}$-category if and only if the data (θ, i, k) defines a monad μ in $\mathsf{Sp}(\mathbf{T})$. *p. 259*

In $\mathcal{E}$, we have the following characterising relations for $i \circ \theta$, $\theta \circ i$, $k \circ \theta$, and $\theta \circ k$:

$$\begin{aligned} v_2 \cdot (i \circ \theta) &= Ti \cdot u'_2 & v_1 \cdot (i \circ \theta) &= u'_1 \\ v_2 \cdot (\theta \circ i) &= u_2 & v_1 \cdot (\theta \circ i) &= i \cdot u_1 \\ v_2 \cdot (k \circ \theta) &= Tk \cdot w_2 & v_1 \cdot (k \circ \theta) &= v_1 \cdot w_1 \\ v_2 \cdot (\theta \circ k) &= w'_2 & v_1 \cdot (\theta \circ k) &= k \cdot w'_1. \end{aligned} \tag{3}$$

Using (1), (2), and (3), we thus deduce the following relations in $\mathcal{E}$:

$$\begin{aligned} v_2 \cdot (i \circ \theta) \cdot l &= Ti \cdot a & v_1 \cdot (i \circ \theta) \cdot l &= \mathrm{id}_\pi \\ v_2 \cdot (\theta \circ i) &= I\pi & v_1 \cdot (\theta \circ i) \cdot r &= i \cdot b \\ v_2 \cdot (k \circ \theta) &= Tk \cdot w_2 & v_1 \cdot (k \circ \theta) &= v_1 \cdot w_1 \\ v_2 \cdot (\theta \circ k) \cdot s &= K\pi \cdot Tv_2 \cdot w_2 & v_1 \cdot (\theta \circ k) \cdot s &= k \cdot w_1. \end{aligned}$$

If we combine these relations with the characterising relations (2), (3), (4), (5), (6) from §I.1, then we deduce the relations

$$\begin{aligned} i_1 &= (i \circ \theta) \cdot l & i_2 &= (\theta \circ i) \cdot r \\ k_1 &= (\theta \circ k) \cdot s & k_2 &= k \circ \theta. \end{aligned}$$

The "identity" and "associativity" conditions (7) and (8) from §I.1 can then be written as

$$\begin{aligned} k \cdot (i \circ \theta) \cdot l = \mathrm{id}_\pi &= k \cdot (\theta \circ i) \cdot r \\ k \cdot (\theta \circ k) \cdot s &= k \cdot (k \circ \theta) \end{aligned}$$

and express the commutativity of the diagrams defined, above, by μ, which are extracts of the following diagram in $\mathsf{Sp}(\mathbf{T})$:

$$\begin{tikzcd}
& & \theta \circ \mathrm{id}_e \arrow[dr, "\theta\circ i"] & & (\theta\circ\theta)\circ\theta \arrow[dl, "k\circ\theta"'] \arrow[dd, "s(\theta,\theta,\theta)"] \\
\mathrm{id}_e \arrow[r, "i"] & \theta \arrow[ur, "r(\theta)"] \arrow[dr, "l(\theta)"'] & & \theta\circ\theta \arrow[ll, "k"'] & \\
& & \mathrm{id}_e \circ \theta \arrow[ur, "i\circ\theta"'] & & \theta\circ(\theta\circ\theta) \arrow[ul, "\theta\circ k"]
\end{tikzcd}$$

which finishes the proof. $\square$ p. 260

If $\mathscr{E}$ satisfies the same hypotheses, we denote by $\mathsf{Sp}(\mathscr{E})$ the pseudo-category (which is really a bicategory) $\mathsf{Sp}(\mathrm{id}_{\mathscr{E}}, \mathrm{id}_{\mathscr{E}}, \mathrm{id}_{\mathscr{E}})$; its morphisms are simply called *spans in* $\mathscr{E}$. We denote by $\circ$ its first law, and by $\cdot$ its second, and we denote a morphism of spans and the underlying morphism in $\mathscr{E}$ by the same notation. We will now interpret a $\mathbf{T}$-category in $\mathsf{Sp}(\mathscr{E})$ instead of $\mathsf{Sp}(\mathbf{T})$. If we had defined the notion of a pseudo-triple on a pseudo-category (for example, we would see that $\mathbf{T}$ induces a pseudo-triple on $\mathsf{Sp}(\mathscr{E})$), then we could define the notion of pseudo-algebra in full generality. We will content ourselves with the following definition: we say that $(\theta, \tilde{i}, \tilde{k})$ is a $\mathbf{T}$-pseudo-algebra on e if $\theta\colon Te \to e$ is a span in $\mathscr{E}$ (and thus defined by morphisms $b\colon \pi \to e$ and $a\colon \pi \to Te$ in $\mathscr{E}$), if

$$\begin{aligned} \tilde{i}&\colon \mathrm{id}_e \to \theta \circ Ie \\ \tilde{k}&\colon \theta \circ T\theta \to \theta Ke \end{aligned}$$

are 2-morphisms in $\mathsf{Sp}(\mathscr{E})$, where we set $T\theta = (Tb, Ta)$ and identify every morphism $x\colon e' \to e''$ with the span $(x, \mathrm{id}_{e'})\colon e' \to e''$, and finally if the following "coherence axioms" are satisfied:

$$(\tilde{k} \circ ITe) \cdot (\tilde{i} \circ \theta) = \mathrm{id}_\theta = (\tilde{k} \circ TIe) \cdot (\theta \circ T\tilde{i}) \tag{1}$$

$$(\tilde{k} \circ KTe) \cdot (\tilde{k} \circ TT\theta) = (\tilde{k} \circ TKe) \cdot (\theta \circ T\tilde{k}). \tag{2}$$

Proposition II.3.16. *The data of a* $\mathbf{T}$*-category on* e *is equivalent to the data of a* $\mathbf{T}$*-pseudo-algebra on* e*.*

Proof. Let $\Theta\colon Te \to e$ be a span, with $\theta = (b, a)$; let (u_1'', u_2'') be the canonical fibre product of (a, Ie); then $\theta \circ Ie\colon e \to e$ is defined by the pair $(b \cdot u_1'', u_2'')$. A morphism $i\colon \mathrm{id}_e \to \theta \circ Ie$ in $\mathsf{Sp}(\mathscr{E})$ is defined by a morphism $\tilde{i}\colon e \to \pi'$ in $\mathscr{E}$ (see **Figure**) such that

$$\begin{aligned} b \cdot u_1'' \cdot \tilde{i} &= \mathrm{id}_e \\ u_2'' \cdot \tilde{i} &= \mathrm{id}_e. \end{aligned}$$

Set $i = u_1'' \cdot \tilde{i}$; we have that

$$\begin{aligned} b \cdot i &= \mathrm{id}_e \\ a \cdot i &= Ie \cdot u_2'' \cdot \tilde{i} = Ie \end{aligned}$$

and so (b, a, i) is a pointed $\mathbf{T}$-graph on e. Conversely, this data determines a crochet $\tilde{i}\colon e \to \pi$ that defines the desired morphism $\tilde{i}$ of spans.

TO-DO figure p. 261

Let (v_1, v_2) and (v_1', v_2') be the canonical fibre products of (a, Tb) and (a, Ke) respectively; then $\theta \circ T\theta$ and $\theta \circ Ke\colon TTe \to e$ are defined by the pairs $(b \cdot v_1, Ta \cdot v_2)$ and $(b \cdot v_1', v_2')$ respectively. A morphism of spans $\tilde{k}\colon \theta \circ T\theta \to \theta \circ Ke$ is defined by a morphism $\tilde{k}\colon \pi_2 \to \pi''$ in $\mathscr{E}$ such that

$$\begin{aligned} b \cdot v_1' \cdot \tilde{k} &= b \cdot v_1 \\ v_2' \cdot \tilde{k} &= Ta \cdot v_2. \end{aligned}$$

Set $k = v_1' \cdot \tilde{k}$; we have that $b \cdot k = b \cdot v_1$, and

$$\begin{aligned} a \cdot k &= a \cdot v_1' \cdot \tilde{k} \\ &= Ke \cdot v_2' \cdot \tilde{k} \\ &= Ke \cdot Ta \cdot v_2 \end{aligned}$$

which expresses axiom (4) of §I.1. Conversely, if k satisfies (4) then it determines a crochet $\tilde{k}$ that defines a morphism of spans $\tilde{k}\colon \theta \circ T\theta \to \theta \circ Ke$. Thus it remains only to show that (b, a, i, k) is a $\mathbf{T}$-category, i.e. that it satisfies axioms (7) and (8) of §I.1, if and only if the coherence axioms (1) and (2) above are satisfied, which is a routine verification. $\square$ *p. 262*

Remark. By applying the above to $\mathbf{T}$-preorders, we reprove Proposition I.2.4.

II.4 Pseudo-monoidal and fibration category of $\mathsf{Gr}(\mathbf{T})$

TO-DO think about section title translation

A *pseudo-monoidal category* (or a *pseudo-monoid*) is a pseudo-category $\overline{\mathscr{D}}$ with only one object, which we almost always denote by 1. If $\mathscr{D}$ is the category $\mathrm{Hom}_{\overline{\mathscr{D}}}(1,1)$, then its composition law, which coincides with the "second law" of $\overline{\mathscr{D}}$ is denoted by $\cdot$ and the "first law" of $\mathscr{D}$ **typo?** $\overline{\mathscr{D}}$ is generally denoted by the symbol $\circ$. We often write $\overline{\mathscr{D}} = (\mathscr{D}, 1, \circ)$, and the symbols r, l, and s from axioms (5) and (6) of §II.1 are always used in the same way; this is why they are easily understood even when denoting $\overline{\mathscr{D}}$ as a triplet. Similarly, a pseudo-functor $\overline{F}\colon \overline{\mathscr{D}} \to \overline{\mathscr{D}}'$ between pseudo-monoidal categories is determined by a functor $F\colon \mathscr{D} \to \mathscr{D}'$ between the underlying categories that "commutes" with the first laws, i.e. the morphisms u and c from axioms (3') and (4') are always denoted in the same way: $u\colon 1 \to F1$ is a morphism in $\mathscr{D}'$, whereas $c\colon \bigcirc \cdot (F \times F) \to F \cdot \bigcirc$, where we denote by $\bigcirc\colon \mathscr{D} \times \mathscr{D} \to \mathscr{D}$ the "first law" functor:

$$\bigcirc(m', m) = m' \circ m$$

for every pair of morphisms in $\mathscr{D}$, for any pseudo-monoidal category $\overline{\mathscr{D}}$. All this data should satisfy "the coherence axioms" (7), (8), (5'), and (6') of §II.1.

Let $\mathbf{T}$ be a triple on a category $\mathscr{E}$ endowed with canonical fibre products. For every $e \in |\mathscr{E}|$, we denote by $\mathsf{Gr}(\mathbf{T})/e$ the category $\mathrm{Hom}_{\mathsf{Sp}(\mathbf{T})}(e, e)$; this is a subcategory of $\mathsf{Gr}(\mathbf{T})$, and underlies a pseudo-monoidal category $\overline{\mathsf{Gr}(\mathbf{T})/e} = (\mathsf{Gr}(\mathbf{T})/e, 1, \circ)$ which is obtained by "restriction" of the structure of $\mathsf{Sp}(\mathbf{T})$: it is actually the dual (for just the $\circ$ law) of a sub-pseudo-category of $\mathsf{Sp}(\mathbf{T})$.

For every morphism $f\colon e \to e'$ in $\mathscr{E}$, we define a functor

$$\begin{aligned} f^*\colon \mathsf{Gr}(\mathbf{T})/e' &\to \mathsf{Gr}(\mathbf{T})/e \\ m &\mapsto f^* m \end{aligned}$$

(see **Proposition I.5.6**).

Proposition II.4.16. *f^* is the underlying functor of a pseudo-functor, denoted by $\overline{f^*}\colon \overline{\mathsf{Gr}(\mathbf{T})/e'} \to \overline{\mathsf{Gr}(\mathbf{T})/e}$, for which the morphisms u and c are further uniquely determined* (but are not necessarily isomorphisms, even if $\mathbf{T}$ is cartesian). *p. 263*

Proof. If θ and θ' are $\mathbf{T}$-graphs on e, then the construction of the morphism $f^*(\theta' \circ \theta)$ by projective limits (which reduces to official fibre products) implies the existence of a unique morphism

$$c(\theta', \theta) \colon f^*\theta' \circ f^*\theta \to f^*(\theta' \circ \theta)$$

in $\mathsf{Gr}(\mathbf{T})/e$ determined by a crochet $c(\theta', \theta) \colon \pi'' \to \pi'$ in $\mathcal{E}$, if π'' and π' are the objects of morphisms of $\mathbf{T}$-graphs

$$f^*\theta' \circ f^*\theta \quad \text{and} \quad f^*(\theta' \circ \theta)$$

respectively. We determine a unique morphism $u \colon 1 \to f^*1$ in an analogous way. We will show that these morphisms satisfy the coherence axioms (5') and (6'). In fact, we will limit ourselves to showing, for example, that the following diagram, for any $\mathbf{T}$-graph $\theta = (b, a)$ over e, commutes:

$$\begin{tikzcd}
f^*(1 \circ \theta) & f^*\theta \arrow[l, "f^*l\theta"'] \arrow[d, "lf^*\theta"] \\
f^*1 \circ f^*\theta \arrow[u, "c(1,\theta)"] & 1 \circ f^*\theta \arrow[l, "u \circ f^*\theta"]
\end{tikzcd}$$

which reduces to showing that we have a commutative diagram with the underlying morphisms in $\mathcal{E}$, by using the projective limit property of $f^*(1 \circ \theta)$. We denote by the same symbol a morphism in $\mathsf{Gr}(\mathbf{T})$ and the underlying morphism in $\mathcal{E}$ which it defines.

Let $(b', a') = f^*\theta$ and $(\overline{b}, \overline{a}) = f^*(1 \circ \theta)$; we have projective limits (where the canonical projections are represented by dashed arrows)

$$\begin{tikzcd}
Te & Te' \arrow[l, "Tf"'] \\
\pi \arrow[u, "a"] \arrow[d, "b"'] & \pi' \arrow[l, dashed, "f'"'] \arrow[u, "a'"'] \arrow[d, "b'"] \\
e & e' \arrow[l, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Te & Te' \arrow[l, "Tf"'] \\
\pi_1 \arrow[u, "Ke \cdot Ta \cdot v_2"] \arrow[d, "v_1"'] & \pi \arrow[l, dashed, "\overline{f}"'] \arrow[u, "\overline{a}"'] \arrow[d, "\overline{b}"] \\
e & e' \arrow[l, "f"]
\end{tikzcd}$$

and where (v_1, v_2) is the fibre product of (Ie, Tb) which is used to form the composite $1 \circ \theta = (v_1, Ke \cdot Ta \cdot v_2)$. *p. 264*

We are thus led to prove the following equalities:

$$\overline{a} \cdot f^*l\theta = \overline{a} \cdot c(1, \theta) \cdot (u \circ f^*\theta) \cdot lf^*\theta \tag{1}$$

$$\overline{f} \cdot f^*l\theta = \overline{f} \cdot c(1, \theta) \cdot (u \circ f^*\theta) \cdot lf^*\theta \tag{2}$$

$$\overline{b} \cdot f^*l\theta = \overline{b} \cdot c(1, \theta) \cdot (u \circ f^*\theta) \cdot lf^*\theta. \tag{3}$$

We set $f^*1 = (b_\circ, a_\circ)$ and let $f_\circ$ be the third canonical projection that defines f^*1:

$$
\begin{tikzcd}
Te & Te' \arrow[l, "Tf"'] \\
e \arrow[u, "Ie"] \arrow[d, "\mathrm{id}_e"'] & \pi_\circ \arrow[l, "f_\circ"'] \arrow[u, "a_\circ"'] \arrow[d, "b_\circ"] \\
e & e' \arrow[l, "f"]
\end{tikzcd}
$$

We first obtain a morphism g in $\mathscr{E}$ such that, denoting by $(v_{1\circ}, v_{2\circ})$ the fibre product of $(a_\circ, Tb')$, we have

$$f^*1 \circ f^*\theta = (b_\circ \cdot v_{1\circ}, Ke' \cdot Ta' \cdot v_{2\circ})$$

and

$$\begin{aligned} v_1 \cdot g &= f_\circ \cdot v_{1\circ} \\ v_2 \cdot g &= Tf' \cdot v_{2\circ}. \end{aligned} \tag{4}$$

The morphism u in $\mathscr{E}$ is characterised by the relations

$$\begin{aligned} b_\circ \cdot u &= \mathrm{id}_{e'} \\ f_\circ \cdot u &= f \\ a_\circ \cdot u &= Ie' \end{aligned} \tag{5}$$

and so, if (v_1', v_2') is the fibre product of (Ie', Tb'), we have

$$1 \circ f^*\theta = (v_1', Ke' \cdot Ta' \cdot v_2')$$

and

$$\begin{aligned} u \cdot v_1' &= v_{1\circ} \cdot (u \circ f^*\theta) \\ v_2' &= v_{2\circ} \cdot (u \circ f^*\theta). \end{aligned} \tag{6}$$

By the definition of the morphism $l\theta$, we have

$$\begin{aligned} v_1 \cdot l\theta &= b \\ v_2 \cdot l\theta &= I\pi \end{aligned}$$

whence the relations

$$\begin{aligned} \overline{a} \cdot f^*l\theta &= a' \\ \overline{f} \cdot f^*l\theta &= l\theta \cdot f' \\ \overline{b} \cdot f^*l\theta &= b'. \end{aligned} \tag{7}$$

Similarly, the morphism $lf^*\theta$ is characterised by *p. 265*

$$\begin{aligned} v_1' \cdot lf^*\theta &= b' \\ v_2' \cdot lf^*\theta &= I\pi'. \end{aligned} \tag{8}$$

We also have that

$$l\theta \cdot f' = g \cdot (u \circ f^*\theta) \cdot lf^*\theta \tag{9}$$

(the "naturality" of l), since

$$\begin{aligned}
v_1 \cdot g \cdot (u \circ f^*\theta) &\underset{(4)}{=} f_\circ \cdot v_{1\circ} \cdot (u \circ f^*\theta) \\
&\underset{(6)}{=} f_\circ \cdot u \cdot v_1' \\
&\underset{(5)}{=} f \cdot v_1'
\end{aligned}$$

and

$$v_2 \cdot g \cdot (u \circ f^*\theta) \underset{(4)}{=} Tf' \cdot v_{2\circ} \cdot (u \circ f^*\theta) \underset{(6)}{=} Tf' \cdot v_2'.$$

Finally, by the definition of $c(1, \theta)$, we have that

$$\begin{aligned}
\overline{a} \cdot c(1, \theta) &= Ke' \cdot Ta' \cdot v_{2\circ} \\
\overline{f} \cdot c(1, \theta) &= g \\
\overline{b} \cdot c(1, \theta) &= b_\circ \cdot v_{1\circ}.
\end{aligned} \tag{10}$$

We have that

$$\begin{aligned}
\overline{a} \cdot f^*l\theta &\underset{(7)}{=} a' \\
&= Ke' \cdot Ta' \cdot I\pi' \\
&\underset{(8)}{=} Ke' \cdot Ta' \cdot v_2' \cdot lf^*\theta \\
&\underset{(6)}{=} Ke' \cdot Ta' \cdot v_{2\circ} \cdot (u \circ f^*\theta) \cdot lf^*\theta \\
&\underset{(10)}{=} \overline{a} \cdot c(1, \theta) \cdot (u \circ f^*\theta) \cdot lf^*\theta
\end{aligned}$$

which proves (1). We have that

$$\begin{aligned}
\overline{f} \cdot f^*l\theta &\underset{(7)}{=} l\theta \cdot f' \\
&\underset{(9)}{=} g \cdot (u \circ f^*\theta) \cdot lf^*\theta \\
&\underset{(10)}{=} f \cdot c(1, \theta) \cdot (u \circ f^*\theta) \cdot lf^*\theta
\end{aligned}$$

which proves (2). Finally,

$$\begin{aligned}
\overline{b} \cdot f^*l\theta &\underset{(7)}{=} b' \\
&\underset{(8)}{=} v_1' \cdot lf^*\theta \\
&\underset{(5)}{=} b_\circ \cdot u \cdot v_1' \cdot lf^*\theta \\
&\underset{(6)}{=} b_\circ \cdot v_{1\circ} \cdot (u \circ f^*\theta) \cdot lf^*\theta \\
&\underset{(10)}{=} \overline{b} \cdot c(1, \theta) \cdot (u \circ f^*\theta) \cdot lf^*\theta
\end{aligned}$$

which proves (3). $\square$

Relative to a universe denoted by $|\mathsf{Set}|$, we denote by Set, Cat, etc. the categories of sets, of categories, etc. (respectively). We denote by $\mathsf{Cat}^\sharp$ the category whose objects are the pseudo-monoidal categories whose underlying category is in Cat and whose morphisms are the pseudo-functors. Relative to a universe denoted by $|\widehat{\mathsf{Set}}|$, we denote by $\widehat{\mathsf{Set}}$, $\widehat{\mathsf{Cat}}$, $\widehat{\mathsf{Cat}}^\sharp$, etc. the analogous categories. If $\mathcal{E}$ is an element of $|\widehat{\mathsf{Cat}}|$, then the previous Proposition shows us how, to a morphism $f\colon e \to e'$ in $\mathcal{E}$, we can associate a morphism p. 266

$$\overline{f^*}\colon \overline{\mathsf{Gr}(\mathbf{T})/e'} \to \overline{\mathsf{Gr}(\mathbf{T})/e}$$

in $\widehat{\mathsf{Cat}}^\sharp$. We define a *bifunctor* to be a pseudo-functor (F, u, c) such that u and c are natural equivalences (and not a pseudo-functor between bicategories, as in [Be]).

Proposition II.4.18. *The correspondence that associates to each morphism f in $\mathcal{E}$ the pseudo-functor $\overline{f^*}$ defines a bifunctor $\mathcal{E}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}^\sharp$.*

Proof. We leave this simple proof as an exercise; it uses the fact that u and c are crochets of projective limits. $\square$

III Examples

p. 267

In the first two examples that follow (§III.2 and **??**), the triple $\mathbf{T}$ possesses properties that simply its study. These properties ensure not only that the functor $\mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T})$ is tripleable (**Proposition II.1.19**), but also that they permit a simple construction of the adjoint (**scolium of Proposition III.2.21**).

III.1 Free T-categories (particular case)

Let $\mathbf{T}$ be a triple on a category $\mathcal{E}$ admitting finite fibre products and endowed with a canonical choice of these fibre products. Let $\mathcal{X}$ be one of the categories $\mathsf{Alg}(\mathbf{T})$, $\mathsf{Ord}(\mathbf{T})$, $\mathsf{Cat}(\mathbf{T})$, or $\mathsf{Gr}(\mathbf{T})$, and $e \in |\mathcal{E}|$; we denote by $\mathcal{X}/e$ the subcategory of $\mathcal{X}$ whose objects are the objects $\overline{e}$ of $\mathcal{X}$ such that $U\overline{e} = e$, where $U\colon \mathcal{X} \to \mathcal{E}$ is the usual forgetful functor (see the sequence $(*)$ at the start of §I.3), and whose morphisms are the morphisms $\overline{f}\colon \overline{e} \to \overline{e}'$ in $\mathcal{X}$ such that $U\overline{f} = \mathrm{id}_e$. Note that $\mathsf{Alg}(\mathbf{T})/e$ is a discrete category and $\mathsf{Ord}(\mathbf{T})/e$ is a preorder.

We say that $\mathbf{T} = (T, I, K)$ is a *strongly cartesian* triple if it is cartesian (§II.2) and further satisfies the following conditions:

(c) $\mathcal{E}$ admits finite and countable sums, and T commutes with these sums.

(d) The sums in $\mathcal{E}$ are universal (§0.1) and commute with fibre products.

Proposition III.1.19. *If $\mathbf{T}$ is a strongly cartesian triple, then the forgetful functor $\mathbf{T} \to \mathsf{Gr}(\mathbf{T})$ is tripleable.*

Proof. Let $U\colon \mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T})$ be the usual forgetful functor; we will define an adjoint functor $F\colon \mathsf{Gr}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T})$ by using **Proposition A.1** in the appendix. For every $e \in |\mathcal{E}|$, by §II.2 and the terminology of §II.4,

$$\overline{\mathsf{Gr}(\mathbf{T})/e} = (\mathsf{Gr}(\mathbf{T})/e, 1, \circ)$$

is a monoidal category, and, by §II.3, $\mathsf{Cat}(\mathbf{T})/e$ is isomorphic to the category of monads $\mathsf{Mon}(\overline{\mathsf{Gr}(\mathbf{T})/e})$ (see the Appendix) and its forgetful functor to $\mathsf{Gr}(\mathbf{T})/e$ agrees with a restriction of U. To show that the functor $\mathsf{Cat}(\mathbf{T})/e \to \mathsf{Gr}(\mathbf{T})/e$ admits an adjoint, it thus suffices to show that $\overline{\mathsf{Gr}(\mathbf{T})/e}$ is countably distributive (**Proposition A.1**). The existence of finite and countable sums follows from hypothesis (c) and from the "creativity" of the functor $\mathsf{Gr}(\mathbf{T})/e \to \mathcal{E}$ that, to each $\mathbf{T}$-graph θ over e, associates the object of the morphisms π and θ. We show that $\overline{\mathsf{Gr}(\mathbf{T})/e}$ is countably distributive by using conditions (c) and (d), along with the fact that, if we have an equality of the form $C' \cdot C = C''$ in $\mathcal{E}$ and if C' and C'' are cartesian natural transformations (§II.2), then C is also cartesian. The forgetful functor $U_e\colon \mathsf{Cat}(\mathbf{T})/e \to \mathsf{Gr}(\mathbf{T})/e$ is thus tripleable; we denote its adjoint by $F_e\colon \mathsf{Gr}(\mathbf{T})/e \to \mathsf{Cat}(\mathbf{T})/e$. We then define $|F|$ by the extension of the maps $|F_e|$, where e runs over $|\mathcal{E}|$. *p. 268*

If $\mathbf{T}$ is strongly cartesian, then so too is the triple $\mathbf{T^2}$ (§I.1), and the forgetful functor $\hat{U}\colon \mathsf{Cat}(\mathbf{T^2}) \to \mathsf{Gr}(\mathbf{T^2})$ can be identified with the functor

$$U^{\mathbf{2}}\colon \mathsf{Cat}(\mathbf{T^2}) \to \mathsf{Gr}(\mathbf{T^2}).$$

The way in which the adjoint of **Proposition A.1** is constructed leads us to define $|F^{\mathbf{2}}|$ as the extension of the maps $|\hat{F}_f|$, where f runs over $|\mathcal{E}^{\mathbf{2}}|$, and where we denote by $\hat{F}_f$ the adjoint of the forgetful functor

$$\hat{U}_f\colon \mathsf{Cat}(\mathbf{T^2})/f \to \mathsf{Gr}(\mathbf{T^2})/f.$$

These conditions thus allow us to construct a functor

$$F\colon \mathsf{Gr}(\mathbf{T}) \to \mathsf{Cat}(\mathbf{T})$$

and this functor is an adjoint of $U\colon \mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T})$; indeed, it suffices to note that the same proof allows us to construct the natural transformations I' and K' of the desired triple, which we denote by

$$\mathbf{T}' = (T', I', K')$$

where $T' = UF$.

It remains to show that the Eilenberg–Moore functor

$$E\colon \mathsf{Cat}(\mathbf{T}) \to \mathsf{Alg}(\mathbf{T}')$$

is an isomorphism. The triple $\mathbf{T}'_e$ induced by the adjoint pair (F_e, U_e) can be obtained by restriction of the triple $\mathbf{T}'$ to $\mathsf{Gr}(\mathbf{T})/e$, and, since the associated Eilenberg–Moore functor $E_e\colon \mathsf{Cat}(\mathbf{T})/e \to \mathsf{Alg}(\mathbf{T}'_e)$ is an isomorphism, we immediately deduce that $|E|$ is a bijection. By an analogous reasoning, we can show that $|E^{\mathbf{2}}|$ is a bijection, since there is a bijection between $\mathbf{T}'^{\mathbf{2}}$-algebras and morphisms between $\mathbf{T}'$-algebras. This proves that E is an isomorphism. ☐ *p. 269*

A morphism in $\mathcal{E}^{\mathbf{2}}$ is said to be *cartesian* if it is cartesian in the sense of §II.2 (such a morphism being a natural transformation). We say that an inductive cone in $\mathcal{E}^{\mathbf{2}}$ is cartesian if its canonical injections are all cartesian.

Lemma. *If sums in $\mathcal{E}$ are universal, then $\mathcal{E}$ satisfies the following property:*

(e) the canonical injections of sums in $\mathcal{E}$ are monomorphisms.

If $\mathcal{E}$ satisfies (d) (and thus (e)), then every finite or countable sum in $\mathcal{E}^{\mathbf{2}}$ is a cartesian cone (if $\mathcal{E}$ admits a final object, then this is true for every sum), and every crochet of a sum formed from a cartesian cone in $\mathcal{E}^{\mathbf{2}}$ is cartesian.

Proof. The first part is an interesting exercise on the language of categories. We then show, using condition (e), that, if a sum in $\mathcal{E}^{\mathbf{2}}$ is a monomorphism (resp. an epimorphism) in $\mathcal{E}$, then each of its terms is a monomorphism (resp. an epimorphism) in $\mathcal{E}$. We subsequently show that the "codiagonal" natural transformations are cartesian by using the universality of sums, and we finish the proof by using the last part of condition (d). □

Proposition III.1.20. *If $\mathbf{T}$ is a strongly cartesian triple, then so too is the triple induced by the tripleable functor $\mathsf{Cat}(\mathbf{T}) \to \mathsf{Gr}(\mathbf{T})$.*

Proof. If $\mathbf{T}' = (T', I', K')$ is the induced triple on $\mathsf{Gr}(\mathbf{T})$, then the proof of **Proposition A.1** in the appendix shows that, for all $\theta \in |\mathsf{Gr}(\mathbf{T})|$, $T'\theta$ is a countable sum, $I'\theta$ is a canonical injection of this sum, and $K'\theta$ is a crochet of the sum $T'T'\theta$; we thus immediately deduce that $\mathbf{T}'$ is cartesian. The rest of the proposition is proven by using the commutativity of projective limits with themselves (fibre products) and the commutativity of inductive limits with themselves (sums). □ *p. 270*

III.2 $\mathcal{E}$-categories

Suppose that $\mathcal{E}$ is a category endowed with finite fibre products. Consider the identity triple $(\mathrm{id}_{\mathcal{E}}, \mathrm{id}_{\mathcal{E}}, \mathrm{id}_{\mathcal{E}})$ on $\mathcal{E}$, we denote it simply by $\mathcal{E}$ in this section. The category $\mathsf{Cat}(\mathcal{E})$ of $\mathcal{E}$-categories is precisely what is usually referred to by that name in the literature (or "$\mathcal{E}$-structured category" in [Eh]). The axioms given in §I.3 then become:

$$\begin{cases} b \cdot i = \mathrm{id}_e \\ a \cdot i = \mathrm{id}_e \end{cases} \tag{1}$$

$$\begin{cases} v_1 \cdot i_1 = \mathrm{id}_\pi \\ v_2 \cdot i_1 = i \cdot a \end{cases} \tag{2}$$

$$\begin{cases} v_1 \cdot i_2 = i \cdot b \\ v_2 \cdot i_2 = \mathrm{id}_\pi \end{cases} \tag{3}$$

$$\begin{cases} b \cdot k = b \cdot v_1 \\ a \cdot k = a \cdot v_2 \end{cases} \tag{4}$$

$$\begin{cases} v_1 \cdot k_1 = k \cdot w_1 \\ v_2 \cdot k_1 = v_2 \cdot w_2 \end{cases} \tag{5}$$

$$\begin{cases} v_1 \cdot k_2 = v_1 \cdot w_1 \\ v_w \cdot k_2 = k \cdot w_2 \end{cases} \tag{6}$$

$$k \cdot i_1 = \mathrm{id}_\pi = k \cdot i_2 \tag{7}$$

$$k \cdot k_1 = k \cdot k_2 \tag{8}$$

where (v_1, v_2) and (w_1, w_2) are the fibre products of (a, b) and (v_2, v_1) respectively. Analogous remarks can be made for $\mathsf{Gr}(\mathscr{E})$, $\mathsf{Ord}(\mathscr{E})$, ... The category $\mathsf{Alg}(\mathscr{E})$ is isomorphic to $\mathscr{E}$, and the category $\mathsf{Equ}(\mathscr{E})$ defined in §I.3 is precisely the category of equivalence $\mathscr{E}$-relations.

The identity triple $\mathscr{E}$ satisfies, evidently, conditions (a) and (b) of §II.2; it it thus cartesian. For it to be strongly cartesian, it is necessary and sufficient that $\mathscr{E}$ satisfy conditions (c), (d), and thus (e), of §II.1. In this case, the forgetful functor $\mathsf{Cat}(\mathscr{E}) \to \mathsf{Gr}(\mathscr{E})$ is tripleable, and the triple $\mathbf{N}$ thus induced is strongly cartesian (**Proposition 20**).

Let Set be the category of sets associated to a universe (which is identical to $|\mathsf{Set}|$). We denote, in principle, by Cat, Gr, ..., the categories of categories, of graphs, of ... (respectively), with these structures being defined in a "local" manner (§I.1). We then have known injective functors $\mathsf{Cat}(\mathsf{Set}) \to \mathsf{Cat}$, $\mathsf{Gr}(\mathsf{Set}) \to \mathsf{Gr}$, ..., which are equivalences of categories. The disjunction convention made in axiom (2) of §I.1 allows us to identify $\mathsf{Cat}(\mathsf{Set})$ with Cat, $\mathsf{Gr}(\mathsf{Set})$ with Gr, ...; but, although this will be useful, context will make precise whether the notation Cat, Gr, ... refer to the "local" or "global" definitions. Similarly, Ord and Equ denote the categories $\mathsf{Ord}(\mathsf{Set})$ and $\mathsf{Equ}(\mathsf{Set})$. Relative to another universe $|\widehat{\mathsf{Set}}|$, we use the notation $\widehat{\mathsf{Cat}}$, $\widehat{\mathsf{Gr}}$, $\widehat{\mathsf{Ord}}$, $\widehat{\mathsf{Equ}}$, ... etc. *p. 271*

We are going to give a description of a strongly cartesian triple, which we will denote by $\mathbf{N} = (N, I, K)$, induced by the tripleable functor $\mathsf{Cat} \to \mathsf{Set}$; we will do so in "local" terminology, and, furthermore, in an intrinsic way (i.e. not relative to a universe).

If $\mathscr{G}$ is a graph, then $N\mathscr{G}$ is the graph such that $|N\mathscr{G}| = |\mathscr{G}|$ and such that $F\colon e \to e'$ is a morphism in $N\mathscr{G}$ if it is a *path in* $\mathscr{G}$, i.e. if there exists a sequence of objects $(e_n, \ldots, e_1, e_0)$ of $\mathscr{G}$, where $e_0 = e$ and $e_n = e'$, and a sequence of morphism $(f_n, \ldots, f_1)$ in $\mathscr{G}$ such that

$$F = (f_n, \ldots, f_1) \quad \text{and} \quad f_i\colon e_{i-1} \to e_i.$$

Such a sequence is also called a sequence of "consecutive" morphisms. The integer n is called the *length* of the path, for simplicity, the path is denoted simply by F. Note that, if $n = 0$, then this definition implies that $e = e'$ $(= e_0)$ and $F = \varnothing$ is the empty sequence: we denote by id_e this path $\varnothing\colon e \to e$ of length zero. Next,

$$I\mathscr{G}\colon \mathscr{G} \to N\mathscr{G} \quad \text{and} \quad K\mathscr{G}\colon NN\mathscr{G} \to N\mathscr{G}$$

are the morphisms of graphs such that

$$|I\mathscr{G}| = |K\mathscr{G}| = id_{|\mathscr{G}|}$$

and defined on morphisms in the following way: $I\mathscr{G}$ associates, to each morphism $f\colon e \to e'$, the path $(f)\colon e \to e'$ of length 1, denoted sometimes simply by $f\colon e \to e'$; $K\mathscr{G}$

associates, to each path $\phi\colon e \to e'$ in $N\mathcal{G}$, the path $\phi\colon e \to e'$ in $\mathcal{G}$ given by "erasing the parentheses" of ϕ (or by "juxtaposition"). In other words, if $\phi = (F_p, \ldots, F_1)$, where p. 272

$$F_i = (f_{i,n_i}, \ldots, f_{i,1})$$

for $1 \leqslant i \leqslant p$, then

$$F = (f_{p,n_p}, \ldots, f_{p,1}, f_{p-1,n_{p-1}}, \ldots, f_{1,1}).$$

The notation of this triple $\mathbf{N}$ can be confused with that for the set of positive or zero integers; if such a confusion risks arising, then we can always identify this set with $N1$.

We sometimes denote by the same symbol a $\mathbf{T}$-category and its underlying $\mathbf{T}$-graph; similarly, we also denote by $N\mathcal{G}$ the free category structure on the graph $N\mathcal{G}$ defined by the free $\mathbf{N}$-algebra $K\mathcal{G}$; we call these the *category* and *graph of paths in* $\mathcal{G}$ (respectively). We call $\mathbf{N}$ the *path triple.*

Proposition III.2.21. *Let $\mathbf{T} = (T, I, K)$ be a cartesian triple on $\mathcal{E}$ and let $\theta = (b, a, i, k)$ be a $\mathbf{T}$-category on $e \in |\mathcal{E}|$; then*

$$(Tb, Ke \cdot Ta, Ti, Tk)$$

is an $\mathcal{E}$-category on Te.

Proof. $(Tb, Ke \cdot Ta, Ti)$ is a pointed $\mathcal{E}$-graph, since

$$Tb \cdot Ti = Te \quad \text{and} \quad Ke \cdot Ta \cdot Ti = Ke \cdot Ie = Te.$$

Let (v_1, v_2) and (w_1, w_2) be the fibre products associated to θ; since the cartesian morphisms of $\mathcal{E}^{\mathbf{2}}$ are stable under composition, $(Tv_1, K\pi \cdot Tv_2)$ and $(Pw_1, K\pi_2 \cdot Tw_2)$ are the fibre products of

$$(Ke \cdot Ta, Tb) \quad \text{and} \quad (K\pi \cdot Tv_2, Tv_1)$$

respectively. If we replace $(b, a, i, k, i_1, i_2, k_1, k_2)$ by

$$(Tb, Ke \cdot Ta, Ti, Tk, Ti_1, Ti_2, Tk_1, Tk_2)$$

in equations (1) to (8) above, then they remain true. □

This proposition allows us to very noticeably simplify, in practice, the description of the free $\mathbf{T}$-category given by §III.1. If $\theta = (b, a)$ is a $\mathbf{T}$-graph on $e \in |\mathcal{E}|$, then we denote by $\bigcirc^n\theta = (b_n, a_n)$ the $\mathbf{T}$-*graph of paths of length* $n \geqslant 0$, the $\mathbf{T}$-graph obtained by using the "first law" of $\mathsf{Sp}(\mathbf{T})$ (§II.2); we denote by π_n its morphism object. p. 273

Scholium. *If $\mathbf{T}$ is a strongly cartesian triple, then the free $\mathbf{T}$-category $\hat\theta = (\hat b, \hat a, \hat\imath, \hat k)$ generated by a $\mathbf{T}$-graph $\theta = (b, a)$ on $e \in |\mathcal{E}|$ has the same object object e as θ. The morphism object $\hat\pi$ of $\hat\theta$ is the sum of the morphism objects π_n for $n \in \mathbb{N}$ that we define by induction: $\pi_0 = e$, $\pi_1 = \pi$, and π_n for $n \geqslant 2$ is the source of the fibre product*

$$\begin{tikzcd}
\pi'_{n-1} \arrow[d, "b'_{n-1}"'] & \pi_n \arrow[l] \arrow[d] \\
Te & \pi \arrow[l, "a"]
\end{tikzcd}$$

where $\theta' = (b', a')$ *is the* $\mathcal{E}$*-graph on* Te *such that* $b' = Tb$ *and* $a' = Ke \cdot Ta$*, and where* π'_{n-1} *is the morphism object of the* $\mathcal{E}$*-graph* $\bigcirc^n \theta' = (b'_n, a'_n)$*.*

We

Bibliography

[Ba] M. Barr. *Relational algebras.* Lecture Notes in Mathematics 137. Springer, 1969.

[Be] J. Benabou. *Introduction to bicategories.* Lecture Notes in Mathematics 47. Springer, 1967.

[Bu] A. Burroni. “Esquisse des catégories à limites et des quasi-topologies”. *Esquisses Mathématiques* 5 (1970).

[Eh] C. Ehresmann. “Catégories structurées généralisées”. *Cahiers Top. et Géom. dif.* X (1968).

[La] J. Lambek. *Deductive systems and categories.* Lecture Notes in Mathematics 86. Springer, 1969.

[Ma] E. Manes. *A triple theoretic construction.* Lecture Notes in Mathematics 80. Springer, 1969.