Complex analytic manifolds and cohomology

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The global theory of ideals of analytic functions, due to K. Oka [10] and H. Cartan [2, 3, 4], holds not only for domains of holomorphy but also for a larger class of complex analytic manifolds introduced by K. Stein [11], and which notably contains all the smooth analytic submanifolds, of arbitrary dimension p, of the complex space of arbitrary dimension $n > p$.

The fundamental theorems of this theory are nicely expressed in the language of cohomology, which suggests generalisations and provides a useful tool when it comes to using the results. In this conference, we will first give an exposition on the fundamental ideas: that of a *sheaf*, and that of *cohomology* with coefficients in a sheaf. We will then state the fundamental theorems. Finally, we will give some applications to global problems concerning Stein manifolds; other applications will be given in the conference by J.-P. Serre.

1 Sheaves on a topological space[1]

Let X be a topological space. A *sheaf of abelian groups* on X, or simply a *sheaf*, is defined by the data: *p. 42*

1. of a function $x \mapsto \mathscr{F}_x$ that, to each point $x \in X$, associates an abelian group $\mathscr{F}_x$ (which we denote additively);
2. of a (not necessarily separated) topology on the union $\mathscr{F}$ of the sets $\mathscr{F}_x$.

Before giving the axioms that these data must satisfy, we denote by p the map from $\mathscr{F}$ to X that, to each $\alpha \in \mathscr{F}$, associates the point x such that $\alpha \in \mathscr{F}_x$. We impose two axioms:

$\mathbf{F_I}$. The map $\alpha \mapsto -\alpha$ that, to each $\alpha \in \mathscr{F}$ associates the inverse of α in the group $\mathscr{F}_{p\alpha}$, is a continuous map from $\mathscr{F}$ to $\mathscr{F}$. The map $(\alpha,\beta) \mapsto \alpha+\beta$ defined on the set $\mathscr{G}$ of pairs (α,β) such that $p(\alpha) = p(\beta)$, and that sends such a pair to the sum $\alpha+\beta$ in the group $\mathscr{F}_{p(\alpha)}$, is a continuous map from the subset $\mathscr{G}$ of $\mathscr{F} \times \mathscr{F}$ to $\mathscr{F}$.

$\mathbf{F_{II}}$. The map p is a *local homeomorphism*, i.e. every element $\alpha \in \mathscr{F}$ admits an open neighbourhood V such that the restriction of p to V is a homeomorphism from V to an open subset of X.

If U is a subset of X, then the collection of the $\mathscr{F}_x$ for $x \in U$, endowed with the topology induced by that of $\mathscr{F}$, is evidently a sheaf on U; we denote it by $\mathscr{F}(U)$, and we call it the sheaf *induced* by $\mathscr{F}$ on U.

Example. *Let G be an abelian group. Let $\mathscr{F}$ be the product $G \times X$, endowed with the product topology (with G being endowed with the discrete topology). The subset $\mathscr{F}_x$ of pairs (g,x), where g runs over G, is evidently endowed with the structure of an abelian group (isomorphic to G). We can immediately verify the axioms (F_I) and (F_{II}). This sheaf is called the* constant sheaf *defined by G, and is also denoted by G.*

Let $\mathscr{F}$ be an arbitrary sheaf on X. We define a *section* of $\mathscr{F}$ over an open $U \subset X$ to be a *continuous* map $s\colon U \to \mathscr{F}$ such that $p \circ s$ is the identity; s is then a homeomorphism from U to its image $s(U)$. Axiom ($\mathrm{F_{II}}$) implies the following: if two sections are equal at a point $x \in U$, then they are equal at all points in a neighbourhood of x. The map s that, to each $x \in U$, associates the identity element $0_x \in \mathscr{F}_x$, is a section (by ($\mathrm{F_I}$)); we call it the *zero section*. The set $\Gamma(U,\mathscr{F})$ of sections of $\mathscr{F}$ over U is endowed with the structure of an abelian group, thanks to ($\mathrm{F_I}$), and the zero section is the identity element of this group.

If U and V are opens such that $V \subset U$, then every section over U induces a section over V. The group $\mathscr{F}_x$ is the inductive limit of the groups $\Gamma(U,\mathscr{F})$ taken over open neighbourhoods U of x. *p. 43*

In practice, a sheaf on X is often defined in the following way: we give abelian groups $\mathscr{F}_U$ attached to certain opens $U \subset X$ that form a fundamental system of opens of the topology of X; and, for every pair (U,V) such that $V \subset U$, we give a homomorphism $f_{VU}\colon \mathscr{F}_U \to \mathscr{F}_V$, such that, for $W \subset V \subset U$, we have that $f_{WU} = f_{WV} \circ f_{VU}$. We then take $\mathscr{F}_x$ to be the inductive limit of the $\mathscr{F}_U$ over the opens U that contain x; and, on the union $\mathscr{F}$ of the $\mathscr{F}_x$, we define the topology $\mathscr{T}$ as follows: for every open U and every $\alpha \in \mathscr{F}_U$, let $[\alpha]$ be the set of images

[1] The notion of *sheaf* was introduced by J. Leray in the study of homological properties of a continuous map. See J. Leray, *Journ. de Math. pures et appliquées* **29**, 1950, pp. 1–139; it is in this work that we find (at the bottom of page 75) a definition of cohomology with coefficients in a sheaf, limited, in truth, to the case of a locally compact space X (and it deals with "compactly supported" cohomology). The definition of sheaves adopted here is a bit different; it is due to Lazard and was explained in [5], where the theory of cohomology with coefficients in a sheaf was developed (exposés XIV to XX).

of α in the $\mathscr{F}_x$ associated to the points $x \in U$; by definition, the subsets $[\alpha]$ of $\mathscr{F}$ constitute a fundamental system of opens of the topology $\mathscr{T}$. Axioms ($\mathrm{F_I}$) and ($\mathrm{F_{II}}$) are satisfied. We have an obvious homomorphism $\mathscr{F}_U \to \Gamma(U, \mathscr{F})$, but this is not necessarily an isomorphism Indeed, two different methods of definition can lead to defining the same sheaf. For the homomorphism $\mathscr{F}_U \to \Gamma(U, \mathscr{F})$ to be an *isomorphism*, it is necessary and sufficient that, for every system of opens U_i with union U, and every system of elements $\alpha_i \in \mathscr{F}_{U_i}$ such that α_i and α_j have the same image in $\mathscr{F}_{U_i \cap U_j}$, there exist a unique $\alpha \in \mathscr{F}_U$ such that $F_{U_i U}(\alpha) = \alpha_i$ for all i.

Example. Let $\mathscr{F}_U$ be the additive group of real-valued (resp. complex-valued) functions defined and continuous on U. If $V \subset U$, then the homomorphism $\mathscr{F}_U \to \mathscr{F}_V$ will be that which sends a function defined on U to its restriction to V. Then $\mathscr{F}_x$ is the additive group of *germs of continuous functions* at the point x; and $\mathscr{F}$ is called the sheaf of germs of real (resp. complex) continuous functions; $\mathscr{F}_U$ is isomorphism to the group of sections of $\mathscr{F}$ over U.

Remark. We have defined sheaves of abelian groups, but it is clear that analogous definitions can be given for any algebraic structure.

2 Subsheaf, homomorphism, quotient sheaf

Let $\mathscr{F}$ be a sheaf on X. Let $\mathscr{G}$ be a subset of $\mathscr{F}$ such that, for all $x \in X$, $G \cap \mathscr{F}_x = \mathscr{G}_x$ is a *subgroup* of $\mathscr{F}_x$. For $\mathscr{G}$, with the topology induced by that of $\mathscr{F}$, to be a sheaf, it is necessary and sufficient that $\mathscr{G}$ be *open* in $\mathscr{F}$ (see condition ($\mathrm{F_{II}}$)). We then say that $\mathscr{G}$ is a *subsheaf* of $\mathscr{F}$.

Let $\mathscr{F}$ and $\mathscr{F}'$ be sheaves on the same space X. We define a homomorphism from $\mathscr{F}$ to $\mathscr{F}'$ to be a *continuous* map f from $\mathscr{F}$ to $\mathscr{F}'$ such that the restriction f_x of f to $\mathscr{F}_x$ is a homomorphism from the group $\mathscr{F}_x$ to the group $\mathscr{F}'_x$. The inverse image of the zero section of $\mathscr{F}'$ (a p. 44
section which is an open in $\mathscr{F}'$) is a subsheaf $\mathscr{G}$ of $\mathscr{F}$, called the *kernel* of the homomorphism f; for each $x \in X$, $\mathscr{G}_x$ is the kernel of f_x. Also, f is an open map; thus the image of $\mathscr{F}$ in $\mathscr{F}'$ is a subsheaf $\mathscr{G}'$ of $\mathscr{F}'$, called the *image* of the homomorphism f; $\mathscr{G}'_x$ is the image of f_x.

It is clear that every homomorphism $f\colon \mathscr{F} \to \mathscr{F}'$ defines, for each open $U \subset X$, a homomorphism of groups of sections $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}')$.

Let $\mathscr{G}$ be a subsheaf of a sheaf $\mathscr{F}$. We define a *quotient sheaf* as follows: let $\mathscr{H}_x$ be the quotient group $\mathscr{F}_x/\mathscr{G}_x$. If $\mathscr{H}$ denotes the union of the $\mathscr{H}_x$, then the maps $\mathscr{F}_x \to \mathscr{H}_x$ define a map from $\mathscr{F}$ to $\mathscr{H}$, which identifies $\mathscr{H}$ with a quotient of $\mathscr{F}$. We endow $\mathscr{H}$ with the quotient topology, which defines $\mathscr{H}$ as a sheaf. This sheaf is denoted $\mathscr{F}/\mathscr{G}$. The map $\mathscr{F} \to \mathscr{F}/\mathscr{G}$ is a homomorphism, whose kernel is the sheaf $\mathscr{G}$. We can describe the sections of the quotient sheaf $\mathscr{F}/\mathscr{G}$ over X: if $s \in \Gamma(X, \mathscr{F}/\mathscr{G})$, then every point $x \in X$ admits an open neighbourhood U such that the section induced by s on U is the image of an element of $\Gamma(U, \mathscr{F})$. Thus X can be covered by opens U_i, and on each U_i we have an element $s_i \in \Gamma(U_i, \mathscr{F})$ such that, on $U_i \cap U_j$, $s_i - s_j$ is a section of the subsheaf $\mathscr{G}$. In general, a section of $\mathscr{F}/\mathscr{G}$ over X is not the image of any section of $\mathscr{F}$ over X. Thus, the sequence of groups and homomorphisms

$$0 \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{F}) \xrightarrow{\varphi} \Gamma(X, \mathscr{F}/\mathscr{G})$$

is evidently an *exact sequence* (i.e. the image of each homomorphism is the kernel of the following homomorphism), but the homomorphism φ is not an epimorphism[2] in general.

[2]A homomorphism $\varphi\colon A \to B$ of abelian groups (or, more generally, of modules) is called an *epimorphism* if φ sends A *onto* B. Recall also the definition of the *cokernel* of φ: it is the quotient of B by the image $\varphi(A)$. There are analogous definitions for homomorphisms of sheaves.

3 Cohomology with coefficients in a sheaf

Let $\mathscr{F}$ be a sheaf on a topological space X. We will define, for every integer $q \geqslant 0$, a cohomology group $\mathrm{H}^q(X,\mathscr{F})$. For every cover $\mathscr{R}$ of X by opens U_i, consider the following: for every integer p. 45 $q \geqslant 0$, we set $p = q+1$, and we associate to each sequence $i_1,\ldots,i_p$ of p indices an element $f_{i_1\ldots i_p} \in \Gamma(U_{i_1} \cap \ldots \cap U_{i_p},\mathscr{F})$, which is zero if $U_{i_1} \cap \ldots \cap U_{i_p}$ is empty; we suppose that $f_{i_1\ldots i_p}$ is an *alternating* function in its indices (in particular, it is zero if the indices are not all distinct). We thus obtain the additive group of *alternating cochains* of the cover $\mathscr{R}$, of degree $q = p-1$, with respect to the sheaf $\mathscr{F}$. We define a *cobordism* operator on this group in the usual way; it increases the degree by 1; whence a cohomology group $\mathrm{H}^q(\mathscr{R},\mathscr{F})$.

If a cover $\mathscr{R}'$ is finer than $\mathscr{R}$, then we have a natural, unique, homomorphism from $\mathrm{H}^q(\mathscr{R},\mathscr{F})$ to $\mathrm{H}^q(\mathscr{R}',\mathscr{F})$. We can thus consider the inductive limit of the $\mathrm{H}^q(\mathscr{R},\mathscr{F})$ over all open covers $\mathscr{R}$; this is, by definition, the group $\mathrm{H}^q(X,\mathscr{F})$. We note that $\mathrm{H}^0(X,\mathscr{F})$ is canonically identified with the group of sections $\Gamma(X,\mathscr{F})$.

Every homomorphism of sheaves $\mathscr{F} \to \mathscr{F}'$ evidently defines homomorphisms $\mathrm{H}^q(X,\mathscr{F}) \to \mathrm{H}^q(X,\mathscr{F}')$. Furthermore, let $\mathscr{G}$ be a subsheaf of $\mathscr{F}$; we can define natural homomorphisms

$$\delta^q : \mathrm{H}^q(X,\mathscr{F}/\mathscr{G}) \to \mathrm{H}^{q+1}(X,\mathscr{G})$$

at least whenever X is paracompact.

We give, for example, the definition of δ^0: we have seen (§2) that an element $\alpha \in \mathrm{H}^0(X,\mathscr{F}/\mathscr{G})$ can be defined by sections $s_i \in \mathrm{H}^0(U_i,\mathscr{F})$ over opens U_i of a suitable cover $\mathscr{R}$ of X; and that $s_i - s_j \in \mathrm{H}^0(U_i \cap U_j,\mathscr{G})$. So set $f_{ij} = s_i - s_j$. The elements f_{ij} define an (alternating) cocycle of degree 1, and thus an element of $\mathrm{H}^1(\mathscr{R},\mathscr{G})$, and thus an element $\beta \in \mathrm{H}^1(X,\mathscr{G})$. We can easily show that this element β is uniquely determined by α, that is, that it is independent of the choice of cover $\mathscr{R}$ and of sections s_i. By definition, $\delta^0(\alpha) = \beta$. For $q > 0$, the definition of δ^q is analogous, but a bit more complicated.

The fundamental property of cohomology is the following[3]: supposing the space X to be paracompact, if $\mathscr{G}$ is a subsheaf of a sheaf $\mathscr{F}$, then *the unbounded sequence of abelian groups and homomorphisms*

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(X,\mathscr{G}) &\to \mathrm{H}^0(X,\mathscr{F}) \to \mathrm{H}^0(X,\mathscr{F}/\mathscr{G}) \\ \xrightarrow{\delta^0} \mathrm{H}^1(X,\mathscr{G}) &\to \ldots \\ \xrightarrow{\delta^{q-1}} \mathrm{H}^q(X,\mathscr{G}) &\to \mathrm{H}^q(X,\mathscr{F}) \to \mathrm{H}^q(X,\mathscr{F}/\mathscr{G}) \\ \xrightarrow{\delta^q} \mathrm{H}^{q+1}(X,\mathscr{G}) &\to \ldots \end{aligned}$$

is an exact sequence. p. 46

The start of this sequence coincides with the sequence

$$0 \to \Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}/\mathscr{G})$$

considered at the end of §2. It thus follows that: for $\Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}/\mathscr{G})$ to be an epimorphism, it suffices that $\mathrm{H}^1(X,\mathscr{G}) = 0$.

[3] See [5], where the property of "exact sequence" was posited as one of the axioms of an axiomatic theory of cohomology with coefficients in a sheaf.

4 Complex analytic manifolds; additive Cousin problem

The well-known definition of a complex manifold of (complex) dimension n (and thus of real dimension $2n$) can be formulated in terms of sheaves, as follows: on the space X, assumed to be separated, we give a subsheaf $\mathscr{O}$ of the sheaf $\mathscr{F}$ of germs of complex-valued continuous functions (§1), and we impose on it the following axiom:

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For each point $x \in X$, there exists an open U containing x and n-many sections f_i of $\mathscr{O}$ over U, zero at the point x, such that

1. the f_i define a homeomorphism from U to an open of the complex space $\mathbb{C}^n$ of (complex) dimension n;
2. the elements of $\mathscr{O}_x$ are exactly the composite functions $F(f_1,\ldots,f_n)$, where F is holomorphic at the origin in $\mathbb{C}^n$.

The systems of n-many sections f_i appearing in these properties are called systems of *local coordinates* at the point x. The sheaf $\mathscr{O}$ is called the sheaf of *germs of holomorphic functions*; the sections of $\mathscr{O}$ over an open U are the *holomorphic functions* on U. We denote by $\mathscr{O}(U)$ the sheaf induced on an open U. We observe that $\mathscr{O}_x$ is an integral ring.

On a complex-analytic manifold X, we define the sheaf $\mathscr{M}$ of *germs of meromorphic functions* as follows: $\mathscr{M}_x$ is the field of fractions of the integral ring $\mathscr{O}_x$; on the union $\mathscr{M}$ of the $\mathscr{M}_x$ we define the following topology: for every pair (f,g) of holomorphic functions on an open U such that g does not take the value 0 on any open non-empty subset of U, let $[f,g]$ be the set of $f_x/g_x \in \mathscr{M}_x$ for $x \in U$ (denoting by f_x, resp. g_x, the image of f, resp. of g, in $\mathscr{O}_x$); the sets $[f,g]$ constitute, by definition, a fundamental system of opens of the topology of $\mathscr{M}$. A *meromorphic function* on an open U is, by definition, a section of $\mathscr{M}$ over U.

It is clear that $\mathscr{O}$ is a subsheaf of $\mathscr{M}$. We interpret the quotient sheaf $\mathscr{M}/\mathscr{O}$ as follows: if $m \in \mathscr{M}_x$, then the class of m in $\mathscr{M}_x/\mathscr{O}_x$ is called the *principal part* of m. A section of $\mathscr{M}/\mathscr{O}$ *p. 47*
is called a *system of principal parts*. Consider the homomorphism $\varphi\colon \Gamma(X,\mathscr{M}) \to \Gamma(X,\mathscr{M}/\mathscr{O})$; to each meromorphic function on X, φ associates a system of principal parts. The classical *additive Cousin problem* (or *first Cousin problem*)[4] consists of characterising, amongst the systems of principal parts on X, those that come from a meromorphic function on X; in other words, characterising the image of the homomorphism φ. To say that the Cousin problem is always solvable is to say that φ is an epimorphism.

By the exact sequence of cohomology (§3), the condition that $\mathrm{H}^1(X,\mathscr{O}) = 0$ is *sufficient* for the additive Cousin problem to always be solvable. We will see that this is most notably the case whenever X is a "Stein manifold" (§7, Theorem B). Before state the general theorems, implying that certain cohomology groups $\mathrm{H}^q(X,\mathscr{F})$ are zero under certain conditions, we must define a new notion: that of "coherent analytic sheaf".

5 Coherent analytic sheaves

Let X be a complex-analytic manifold. An *analytic sheaf* is a sheaf $\mathscr{F}$ such that, for each point x, $\mathscr{F}_x$ is endowed with the structure of a *module* over the ring $\mathscr{O}_x$ (the ring of germs of holomorphic functions at the point x), and such that the map $(f,\alpha) \mapsto f\alpha$, defined on the set $\mathscr{G}$

[4]Cf. [7]. The Cousin problems have given rise to an abundant literature; we restrict ourselves to referring to [9].

of pairs (f,α) such that there exists $x \in X$ with $f \in \mathscr{O}_x$ and $\alpha \in \mathscr{F}_x$, is a *continuous* map from $\mathscr{G} \subset \mathscr{O} \times \mathscr{F}$ to $\mathscr{F}$.

Example. Let $\mathscr{O}^p$ be the direct sum of p-many copies of the sheaf $\mathscr{O}$; an element of $\mathscr{O}_x^p$ is a sequence $(f_1,\ldots,f_p)$ of p-many germs of holomorphic functions at the point x. We define an action of $\mathscr{O}_x$ on $\mathscr{O}_x^p$ by the formula

$$f \cdot (f_1,\ldots,f_p) = (ff_1,\ldots,ff_p)$$

and this defines $\mathscr{O}^p$ as an analytic sheaf.

Another example. Let $\mathscr{I}$ be a subsheaf of $\mathscr{O}$ such that, for each point x, $\mathscr{I}_x$ is an *ideal* of the sheaf $\mathscr{O}_x$; then $\mathscr{I}$ is an analytic sheaf.

Let $\mathscr{F}$ and $\mathscr{F}'$ be analytic sheaves on X. A homomorphism of sheaves $f\colon \mathscr{F} \to \mathscr{F}'$ is said to be *analytic* if, for each point x, the homomorphism $f_x\colon \mathscr{F}_x \to \mathscr{F}'_x$ is compatible with the action of $\mathscr{O}_x$. The kernel, the image, and the cokernel of f are then analytic sheaves. *p. 48*

Definition. We say that an analytic sheaf $\mathscr{F}$ on X is *coherent*[5] if each $x \in X$ admits an open neighbourhood U such that the induced analytic sheaf $\mathscr{F}(U)$ is isomorphic to the cokernel of an analytic homomorphism $f\colon \mathscr{O}^p(U) \to \mathscr{O}^q(U)$ for integers p,q.

In particular, we say that an analytic sheaf is *locally free* if each $x \in X$ admits an open neighbourhood U such that the induced sheaf $\mathscr{F}(U)$ is isomorphic to $\mathscr{O}^q(U)$ for some suitable integer q. Then, every locally free sheaf is coherent.

We can prove the following: *let $\mathscr{F}$ and $\mathscr{F}'$ be coherent analytic sheaves, and f an analytic homomorphism from $\mathscr{F}$ to $\mathscr{F}'$; then the kernel, the image, and the cokernel of f are coherent analytic sheaves*. The proof relies essentially on the theorem of Oka[6], which states the following: for every analytic homomorphism $f\colon \mathscr{O}^p(X) \to \mathscr{O}^q(X)$, every point $x \in X$ admits an open neighbourhood U such that the kernel of the induced homomorphism $\mathscr{O}^p(U) \to \mathscr{O}^q(U)$ is the image of an analytic homomorphism $\mathscr{O}^r(U) \to \mathscr{O}^p(U)$.

We note a necessary and sufficient condition for an analytic subsheaf $\mathscr{G}$ of a coherent analytic sheaf $\mathscr{F}$ to be coherent: for each $x \in X$, there exists an open neighbourhood U of x and a finite number of sections of $\mathscr{F}$ on U such that, for each $y \in U$, $\mathscr{G}_y$ is the sub-$\mathscr{O}_y$-module of $\mathscr{F}_y$ generated by these sections. This criterium notably applies when $\mathscr{F} = \mathscr{O}^q$, and in particular when $\mathscr{F} = \mathscr{O}$; in this latter case, $\mathscr{G}_x$ is an ideal of $\mathscr{O}_x$, and we have a *coherent sheaf of ideals*.

Example of a coherent sheaf of ideals. Let X be a complex-analytic manifold. We defined an *analytic submanifold* of X to be a closed subset V of X such that, for each $x \in V$, there exists an open neighbourhood U of x and a finite number of holomorphic functions f_i on U, such that $y \in V \cap U$ is equivalent to "$y \in U$ and $f_i(y) = 0$". A point $x \in V$ is said to be *regular* if there exists, in the neighbourhood of x in the ambient manifold X, a system of local coordinates $x_1,\ldots,x_n$ that are zero at the point x and such that V is locally defined (on the neighbourhood of x) by the vanishing of some of these coordinates. When all the points of V *p. 49* are regular, we say that V is *regularly embedded* in X. With these definitions, let V be an analytic submanifold of X; it defines a sheaf of ideals on X as follows: at a point $x \in V$, we take the ideal $\mathscr{I}_x$ of $\mathscr{I}_x$ consisting of germs that vanish identically on V on the neighbourhood of x, and at a point $x \notin V$ we take $\mathscr{I}_x = \mathscr{O}_x$. We can show[7] that this sheaf $\mathscr{I}$ (called the sheaf

[5] This definition is more general than that given in [3] and [4]; in the more specific cases treated in [3] and [4], the definitions agree.

[6] See [10, 18 and onwards]; see also [4, 37, Theorem 1]; and [5, Exposé XV, pp. 5–10].

[7] See [4, Theorem 2], and [5, Exposé XVI].

of the submanifold V) is *coherent*; this is evident when V is regularly embedded, but it is true in all cases.

6 Stein manifolds

A Stein manifold is, basically, a complex analytic variety on which there are sufficiently many holomorphic functions. More precisely, it is a complex-analytic manifold (connected or not) that is the countable union of compacts, and that further satisfies the three following conditions:

a. If $x, y \in X$ and $x \neq y$, then there exists a holomorphic function f on X such that $f(x) \neq f(y)$;

b. For every $x \in X$, there exist n-many holomorphic functions on X that induce, in the ring $\mathscr{O}_x$, a system of local coordinates at the point x (where n is the complex dimension of X);

c. The *envelope* (or *hull*) $\hat{K}$ of any compact $K \subset X$ is compact.

Recall the definition of the envelope of a compact K: it is the set $\hat{K}$ of points $x \in X$ such that

$$|f(x)| \leqslant \sup_{y \in K} |f(y)|$$

for all holomorphic f on X.

We can show (using the Baire theorem) that condition (c) is equivalent to the following:

c'. For every infinite sequence S of points in X, not containing any adherent point of X, there exists a holomorphic function on X that is *unbounded* on S.

Examples.

1. Condition (a) shows that a *compact* manifold X of dimension $n > 0$ is never a Stein manifold.

2. For $n = 1$, every *non-compact* connected "Riemann surface" is a Stein manifold: this follows from a thesis of Behnke and Stein [1]. *p. 50*

3. Let X be an open of the complex space $\mathbb{C}^n$. Conditions (a) and (b) are trivially satisfied; condition (c) expresses that X is a *domain of holomorphy* (cf. [6]).

4. Let X be an "étalé domain" in $\mathbb{C}^n$, i.e. a manifold endowed with a map φ into $\mathbb{C}^n$, with φ a local homeomorphism. Condition (b) is trivially satisfied; conditions (a) and (c) imply that X is a domain of holomorphy. Conversely, is every domain of holomorphy (étalé in $\mathbb{C}^n$) a Stein manifold? The answer is positive[8] whenever X is "of finite type" (i.e. if it is impossible to find, in $\mathbb{C}^n$, an open disc A and, in X, an infinity of opens that φ sends homeomorphically to A). The question remains open in the general case.

5. The product $X \times Y$ of two Stein manifolds is evidently a Stein manifold.

[8] Cf. [6], and [5, Exposé IX].

6. Let X be a Stein manifold, and V an analytic submanifold regularly embedded into X (cf. §5). Then V is a Stein manifold: this follows immediately from the definitions. Thus, every analytic submanifold regularly embedded into $\mathbb{C}^n$ is a Stein manifold; in particular, every affine algebraic variety is a Stein manifold.

7. Let X be a Stein manifold, and g a holomorphic function on X that is not identically zero. The set Y of points $x \in X$ such that $g(x) \neq 0$ is a Stein manifold: to show that Y satisfies condition (c), we note that $1/g$ is holomorphic on Y.[9] For example, the variety of the complex linear group in r-many variables is a Stein manifold; thus the variety of any closed subgroup of the complex linear group is a Stein manifold.

7 Statement of the fundamental theorems

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Theorem A. *Let X be a Stein manifold, and $\mathscr{F}$ a coherent analytic sheaf on X. Then, for every point $x \in X$, the image of $\mathrm{H}^0(X, \mathscr{F})$ in $\mathscr{F}_x$ generates $\mathscr{F}_x$ as an $\mathscr{O}_x$-module.*

Theorem B. *Let X be a Stein manifold, and $\mathscr{F}$ a coherent analytic sheaf on X. Then, for every integer $q > 0$, the cohomology groups $\mathrm{H}^q(X, \mathscr{F})$ are zero.*

The proof is too long and delicate to be able to be given here.[10] The proof of Theorem A, and that of Theorem B for the case $q = 1$, in fact constituted the essential object of the thesis [4], at least in the case where X is a domain of holomorphy of finite type, and $\mathscr{F}$ is a coherent subsheaf of $\mathscr{O}^q(X)$. The arguments can easily be transported to the general case of a Stein manifold. The cohomological formulation of Theorem B, and the idea of studying not only the case $q = 1$ but the case of arbitrary $q > 0$, are due to J.-P. Serre.

8 Applications

Theorem 1. *On a Stein manifold X, the additive Cousin problem is always solvable.*[11]

Proof. Indeed, by Theorem B, we have that $\mathrm{H}^1(X, \mathscr{O}) = 0$. □

Theorem 2. *Let X be a Stein manifold, V an analytic submanifold of X, and $\mathscr{I}$ the sheaf of ideals defined by V (§5). Then the holomorphic functions on X that vanish at every point of V generate, at each point $x \in X$, the ideal $\mathscr{I}_x$.*

Proof. Indeed, $\mathscr{I}$ is a coherent sheaf, to which we apply Theorem A. □

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[9] We have a more general result: if, from a Stein manifold, we extract a "divisor" D (i.e. an analytic submanifold of X that, on the neighbourhood of each point $x \in D$, can be defined by a single equation $g_x = 0$, where g_x is holomorphic at the point x and not identically zero), then the *complement manifold $X \setminus D$ is a Stein manifold*. Proving this reduces to proving the following: for all $x \in D$, there exists a function f that is meromorphic on X and holomorphic on $X \setminus D$, and such that $g_x f$ is holomorphic and non-zero at the point x. The existence of such an f follows from Theorem A (§7 below) applied to the subsheaf $\mathscr{F}$ of the sheaf $\mathscr{M}$ of germs of meromorphic functions, defined as follows: $\mathscr{F}_x = \mathscr{O}_x$ if $x \notin D$; if $x \in D$, then $\mathscr{F}_x$ consists of the $\varphi \in \mathscr{M}_x$ such that $g_x \varphi$ is holomorphic. This sheaf is coherent. The proof that has just been sketched is due to J.-P. Serre.

[10] A complete proof was given in [5, Exposé XIX].

[11] Proven for the first time by Oka [9] in the case where X is a univalent domain of holomorphy.

Corollary. *If $x \notin V$, then there exists a function f that is holomorphic on X, zero on V, and such that $f(x) \neq 0$. In other words: the submanifold V can be globally defined by equations (obtained by setting equal to* 0 *some holomorphic functions on X). Furthermore: for every open* p. 52
U of X that is relatively compact, there exists a finite *number of f_i that are holomorphic on X, zero on V, and do not have, in U, any common zero apart from the points of $V \cap U$.*

Theorem 3. *Let X be a Stein manifold, and V an analytic submanifold regularly embedded into X. Then every holomorphic function on the complex-analytic manifold V is induced by a holomorphic function on X.*

Proof. Indeed, let $\mathscr{I}$ be the sheaf of ideals defined by V. By Theorem B, we have that $\mathrm{H}^1(X, \mathscr{I}) = 0$, and so

$$\mathrm{H}^{(}X, \mathscr{O}) \to \mathrm{H}^0(X, \mathscr{O}/\mathscr{I}) \tag{1}$$

is an *epimorphism*. But $\mathscr{O}_x/\mathscr{I}_x$ is zero if $x \notin V$; if $x \in X$, then $\mathscr{O}_x/\mathscr{I}_x$ can be identified with the ring of germs of holomorphic functions on V at the point x, since the point x is regular for V. Thus $\mathrm{H}^0(X, \mathscr{O}/\mathscr{I})$ can be identified with the ring of holomorphic functions on V, and (1) is the homomorphism that, to each holomorphic function on X, associates the function that it induces on V. Whence the theorem. □

Consider the particular case where V is an infinite discrete subset of X: if X is a Stein manifold, then Theorem 3 shows that there exists an f that is holomorphic on X and *takes arbitrary given values on the points of V*. This property evidently strengthens conditions (a) and (c') of Stein manifolds. More generally:

To each point x of a discrete set A, associate an integer $r(x)$. For $x \in A$, let $\mathscr{I}_x$ be the ideal of $\mathscr{O}_x$ consisting of germs whose Taylor expansion, at the point x, has no term of degree $\leqslant r(x)$; for $x \notin A$, set $\mathscr{I}_x = \mathscr{O}_x$. It is immediate that the sheaf $\mathscr{I}$ given by the $\mathscr{I}_x$ is coherent. If $x \notin A$, an element of $\mathscr{O}/\mathscr{I}_x$ is an "expansion of order $r(x)$". We can then state:

Theorem 4. *If X is a Stein manifold, then there exists a function that is holomorphic on X that admits, at each point $x \in A$, an arbitrary given expansion (of arbitrary order $r(x)$).*

Proof. Indeed, to give such a system of expansions is to give an element of $\mathrm{H}^0(X, \mathscr{O}/\mathscr{I})$. But the homomorphism

$$\mathrm{H}^0(X, \mathscr{O}) \to \mathrm{H}^0(X, \mathscr{O}/\mathscr{I})$$

is an epimorphism, since $\mathrm{H}^1(X, \mathscr{I}) = 0$ by virtue of Theorem B. □

Theorem 4, applied to the case where A consists of a single point, strengthens property (b) of the definition of Stein manifolds. p. 53

Remark (due to J.-P. Serre). For a complex-analytic manifold X given by a countable union of compacts to be a Stein manifold, it is necessary and sufficient that X satisfy the following condition:

(S). For every coherent sheaf of ideals $\mathscr{I}$, we have that $\mathrm{H}^1(X, \mathscr{I}) = 0$.

The condition is necessary, by Theorem B. It is sufficient, since (S) implies the validity of the conclusion of Theorem 4, and this implies that X satisfies conditions (a), (b), and (c').

Theorem 5. *Let X be a Stein manifold, and $\mathscr{F}$ a coherent analytic sheaf. If there exist a finite number of sections $u_i \in \mathrm{H}^0(X, \mathscr{F})$ such that, for all $x \in X$, the $\mathscr{O}_x$-module $\mathscr{F}_x$ is generated by the images of the u_i in $\mathscr{F}_x$, then the u_i generate $\mathrm{H}^0(X, \mathscr{F})$ as a module over the ring $\mathrm{H}^0(X, \mathscr{O})$ of holomorphic functions on X.*

Proof. Let p be the number of u_iy the u_i define an analytic homomorphism of sheaves $\mathscr{O}^p \to \mathscr{F}$. By hypothesis, this is an epimorphism. But its kernel is a coherent sheaf; thus, by Theorem B, $\mathrm{H}^0(X, \mathscr{O}^p) \to \mathrm{H}^0(X, \mathscr{F})$ is an epimorphism, and this proves the theorem. □

Example. Take $\mathscr{F} = \mathscr{O}$. To say that the ideal of $\mathscr{O}_x$ generated by the $u_i \in \mathrm{H}^0(X, \mathscr{O})$ is $\mathscr{O}_x$ is equivalent to saying that the holomorphic functions u_i have no common zero in X. Theorem 5 then affirms that there exists an identity of the form

$$1 = \sum c_i u_i \tag{2}$$

with coefficients c_i holomorphic in X.[12] Notably, this result holds when X is an open of $\mathbb{C}^n$ and is a domain of holomorphy. We will show that it does not hold when X is an open of $\mathbb{C}^n$ but is *not* a domain of holomorphy: there then exists a point a on the boundary of X such that every holomorphic function on X extends to a holomorphic function on an open $Y \supset X$ such that $a \in Y$. Take $u_i = x_i - a_i$ (where the x_i are the complex coordinates of a point in $\mathbb{C}^n$, and the a_i are the coordinates of the point a). The u_i do not have a common zero in X, but there is not an identity such as (2), since the c_i, being holomorphic on X, would also be holomorphic at the point a; but the relation (2) cannot be satisfied at the point a.

9 Various extensions; unsolved problems

The additive Cousin problem can be solvable without X necessarily being a Stein manifold. For example, it is solvable for *complex projective space* $\mathbb{P}$, of arbitrary dimension n; indeed, $\mathrm{H}^q(P, \mathscr{O}) = 0$ *for all* $q > 0$. The proof of this result is completely different to that of Theorem B: it follows from a theorem of Dolbeault[13] that if X is a compact Kähler variety then the (complex) vector space $\mathrm{H}^q(X, \mathscr{O})$ is isomorphic to the space of harmonic forms of type $(0, q)$; but, in the case where X is the projective space $\mathbb{P}$, every harmonic form that is not identically zero is of even degree $2p$ and of type (p, p), as follows from the multiplicative structure of the cohomology ring of $\mathbb{P}$ with complex coefficients. *p. 54*

On the other hand, Theorems A and B from §7 can be extended to the following case: let Y be a closed subset of a complex-analytic manifold X; the notion of coherent analytic sheaf can be defined in an obvious way for sheaves on the space Y. We can prove: if Y admits a fundamental system of open neighbourhoods such that each one is a Stein manifold, then Theorems A and B hold for Y (and for every coherent sheaf on Y). For example, take $X = \mathbb{C}^n$ and $Y = \mathbb{R}^n$ (real space embedded into complex space); we can easily see that we find ourselves satisfying the previous conditions. We thus deduce an extension of the theory to *real-analytic submanifolds* of the space $\mathbb{R}^n$; the theorems concern real-analytic coherent sheaves (we can reduce to the case of complex-analytic sheaves by extension of the base field). We obtain, for example, the following result: if V, real-analytic, is regularly embedded into $\mathbb{R}^n$, then every real-analytic function, defined on V, is induced by a real-analytic function on $\mathbb{R}^n$.

Problem. Instead of restricting ourselves to real-analytic submanifolds regularly embedded in $\mathbb{R}^n$, we consider, in general, "real Stein manifolds", i.e. real-analytic (abstract) manifolds that satisfy conditions (a), (b), and (c) of §7, up to replacing the word "holomorphic" by "real-analytic" everywhere. Are there, for real Stein manifolds, analogous theorems to Theorems A and B?

[12] Cf. [3, p. 189] for the case where X is a domain of holomorphy.
[13] [8, Theorem 1].

Another problem. Let X be a (complex) Stein manifold. Given an open $U \subset X$, under what conditions is U a Stein manifold? A *necessary* condition is that every adherent point of U has, in X, an open neighbourhood V such that $V \cap U$ is a Stein manifold (and, on this topic, note that if U and V are Stein opens, then $U \cap V$ is a Stein open). *Is this necessary condition sufficient?* In the particular case where X is a univalent domain of holomorphy of the space $\mathbb{C}^n$, the answer is positive by a theorem of Oka,[14] which has only been proven in the case $n = 2$. *p. 55*

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